Theory of ODEs

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Def$^{\underline{n}}$. $F(s) = \underline{O(s)}$ if $\exists c \in \mathbb{R}$ *s.t.* $|F(s)| \leq cs$ for sufficiently small s.

Lemma. *Cauchy's Law:* $\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2, \quad \alpha, \beta \in \mathbb{R}$

Lemma. *Cauchy-Schwarz:* $x \cdot y \leq |x||y|$

Lemma. *Triangle inequality:* $|x + y| \leq |x| + |y|$

Def$^{\underline{n}}$. $U \subseteq \mathbb{R}^n$, $\underline{C(U)} := \{f : U \to \mathbb{R} : f \text{ continuous}\}$
$\underline{C(U;\mathbb{R}^m)} := \{f : U \to \mathbb{R}^m : f_i \in C(U),\ i = 1, \dots, m\}$
$\underline{||f||_{\infty,u}} := \sup_{x\in\mathbb{R}} |f(x)|$

Def$^{\underline{n}}$. $I \subseteq \mathbb{R}$ –interval $\underline{C^j(I;\mathbb{R}^n)} := \left\{ \begin{array}{c} f : I \to \mathbb{R}^n : j \text{ times differentiable in int}(I) \\ \& \ f^{(k)} \in C(I;\mathbb{R}^n), k = 0, \dots \end{array} \right\}$

Def$^{\underline{n}}$. $U \subseteq \mathbb{R}^n,\ f_j \in C(U;\mathbb{R}^m) \ni f,\ f_j \to f$ uniformly in U if $\|f_j - f\|_{\infty,u} \to 0$ as $j \to \infty$

Def$^{\underline{n}}$. U is compact if $\forall (x_j)_{j\in\mathbb{N}} \subseteq U,\ \exists x_{j_k} \to x \in U$

Def$^{\underline{n}}$. $U \in \mathbb{R}^n,\ U$ is compact iff U is closed & bounded.

Def$^{\underline{n}}$. $\dot{x}(t) = Ax(t) + b(t),\ A \in \mathbb{R}^{n\times n},\ b \in C([0,1];\mathbb{R}^n),\ x_0 \in \mathbb{R}^n,\ x(0) = x_0,\ x : [0,T) \to \mathbb{R}^n$ is a solution if $x \in C([0,T);\mathbb{R}^n)$ & satisfies the above.

Def$^{\underline{n}}$. Operator norm: $A \in \mathbb{R}^{n\times m}, \|A\| := \sup_{\substack{h \in \mathbb{R}^m \\ |h| = 1}} |Ah|$

Def$^{\underline{n}}$. Frobenius norm: $A \in \mathbb{R}^{n\times m},\ \|A\|_{\mathcal{F}} := \left(\sum_{i=1}^n \sum_{j=1}^m |A_{ij}|^2\right)^{1/2}$

Lemma. $\|\cdot\|_{\mathcal{F}}$ *is a norm on* $\mathbb{R}^{n\times m}$

i) $\|A\|_{\mathcal{F}} \geq 0,\ \|A\|_{\mathcal{F}} = 0 \iff A = 0$

ii) $\|\lambda A\|_{\mathcal{F}} = |\lambda|\|A\|_{\mathcal{F}},\ \forall\lambda \in \mathbb{R}$

iii) $\|A + B\|_{\mathcal{F}} \leq \|A\|_{\mathcal{F}} + \|B\|_{\mathcal{F}}$
Additionally:

iv) $|Ah| \leq \|A\|_{\mathcal{F}}|h|,\ h \in \mathbb{R}^m$

v) $\|AB\|_{\mathcal{F}} \leq \|A\|_{\mathcal{F}}\|B\|_{\mathcal{F}},\ \forall A \in \mathbb{R}^{n\times m}, \forall B \in \mathbb{R}^{m\times v}$

Th$^{\underline{m}}$. .

a) Let $A \in \mathbb{R}^{n\times n} \implies \sum_{n=0}^\infty \frac{A^n}{n!}$ *converges in* $\|\cdot\|_{\mathcal{F}}$ *(each component is absolutely convergent).*

b) Let $E(t) := e^{At}$ *then* $E \in C^\infty(\mathbb{R};\mathbb{R}^{n\times n})$ *&* $\dot{E}(t) = AE(t)$.

Prop$^{\underline{n}}$. *M-test:* $\sum_{n=0}^\infty \left\|\frac{t^n A^n}{n!}\right\|_{\mathcal{F}} \leq \sum_{n=0}^\infty m_n \leq +\infty \implies$ *Converges uniformly.*

Prop$^{\underline{n}}$. *<u>Duhamel's Formula</u>: Let $A \in \mathbb{R}^{n\times n},\ b \in C^k([0,T];\mathbb{R}^n)$ & let $x(t) := e^{At}x_0 + \int_0^t e^{A(t-s)}b(s).ds \implies x \in C^{k+1}([0,T);\mathbb{R}^n)$ & solves $\begin{cases} \dot{x} = Ax + b \\ x(0) = x_0 \end{cases}$ $(A \in \mathbb{R}^{n\times n},\ b, x_0 \in \mathbb{R}^n)$*

Lemma. *$f_m \in C^1(I;\mathbb{R}), g \in C(I;\mathbb{R}), \dot{f_m} \to g$ uniformly, $f_m \to f$ pointwise $\implies f \in C^1(I;\mathbb{R})$ if $\dot{f} = g$.*

Prop$^{\underline{n}}$. *<u>M-test</u>: $\sum_{n=0}^{\infty} \|f\|_{\infty,I} < \infty \implies \sum_{n=0}^{m} f_n$ converges uniformly.*

Prop$^{\underline{n}}$. *$\exists$ at most one $x \in C^1([0,T];\mathbb{R}^n)$ solving $\begin{cases} \dot{x} = Ax + b \\ x(0) = x_0 \end{cases}$*

Cor$^{\underline{ly}}$. *Let $A \in \mathbb{R}^{n\times n}$ & $S := \{x \in C^1([0,T];\mathbb{R}^n) : \dot{x} = Ax\} \implies \dim(S) = n$*

Prop$^{\underline{n}}$. *Let $b \in C([0,T],\mathbb{R}^n),\ A \in \mathbb{R}^{n\times n}$, then $\{x \in C^1([0,T],\mathbb{R}^n) : \dot{x} = Ax + b\} = \int_0^t e^{A(t-s)}b(s).ds + S$ $= x_p(t) + \{x_h(t)\}$*

Lemma. *<u>Gronwall's Lemma</u>: $\dot{z} = Az,\ z = x - y,\ z_0 = x_0 - y_0$ so $\dot{\zeta} \le a\zeta,\ \zeta(0) = \zeta_0 \implies \zeta(t) \le e^{at}\zeta_0$, $|x(t) - y(t)| \le e^{\|A\|_{\mathcal{F}} t}|x_0 - y_0|$*

Def$^{\underline{n}}$. $U \subseteq \mathbb{R}^n,\ f : U \to \mathbb{R}^m$, f is a <u>Lipschitz continuous</u> in U if $\exists L \ge 0$ s.t. $|f(x) - f(y)| \le |x - y| \quad \forall x, y \in U$

Def$^{\underline{n}}$. f is <u>locally Lipschitz</u> in U if f is Lipschitz in $K \subseteq U$ for every compact $K \subseteq U$. (K closed & bounded.)

Def$^{\underline{n}}$. U open in $\mathbb{R}^n,\ f : U \to \mathbb{R}^m$ is <u>(Frechet) differentiable</u> at $x \in U$ if $\exists A \in \mathbb{R}^{m\times n}$ s.t. $\frac{f(x+h)-(f(x)+Ah)}{|h|} \to 0$ as $h \to 0$. $\partial f(x) := \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{j=1,\dots,n}^{i=1,\dots,m}$

Prop$^{\underline{n}}$. *<u>Chain Rule</u>: $U \in \mathbb{R}^n$ open, $f \in C^1(U;\mathbb{R}^m) \implies \frac{d}{dt}f(x(t)) = \partial f(x(t))\dot{x}(t)$*

Prop$^{\underline{n}}$. *$U \subseteq \mathbb{R}^n$ open & convex, $f \in C^1(U;\mathbb{R}^m) \implies f$ is Lipscitz in U iff $\|\partial f\|_{\infty,U} < +\infty$ & $|f(x) - f(y)| \le \|\partial f\|_{\infty,U}|x - y|$*

Def$^{\underline{n}}$. $U \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall x, y \in U\ \forall s \in [0,1],\ (1-s)x + sy \in U$.

Prop$^{\underline{n}}$. *$U \subseteq \mathbb{R}^n$ closed & convex, $U = \bigcup_{i=1}^m \bar{U}_i$, U_i <u>open & disjoint.</u> If $\{(1-s)x + sy : s \in [0,1]\} \subset U \implies \exists s_1 < \cdots < s_k$ s.t. $\{(1-s)x + sy : s \in [s_{k-1}, s_k]\} \subseteq \overline{U_{i_k}}$ for some $i_k \in \{1,\dots,m\}$. If $f \in C(U;\mathbb{R}^m) \cap \bigcap_{i=1}^m c^1(U_i;\mathbb{R}^m)$ & $L := \max_{m=1,\dots,m} \|\partial f\|_{\infty,U_i} < \infty \implies f$ is Lipschitz in U & $|f(x) - f(y)| \le L|x - y|$*

Remark. $|f(x) - f(y)| \le \omega(|x - y|)$. ω–modulus of continuity, $\omega : [0,\infty) \to [0,\infty)$
•Lipschitz: $\omega(r) = Lr$, •Hölder: $\omega(r) = cr^\alpha$

Lemma. *Let $(X, \|\cdot\|_X)$ be a Banach (complete, normed, linear) space. $U \subseteq X$ closed, $T : U \to U$ contraction. i.e: $\exists k \in [0,1)$ s.t. $\|T(x) - T(y)\|_X \le k\|x - y\|_X\ \forall x, y \in U \implies \exists! x \in U$ s.t.$T(x) = x$*

Note. X is complete if $(x_n)_{n\in\mathbb{N}} \subseteq X$ Cauchy $\implies \exists x \in X s.t. x_n \to x$

Remark. $X = C([0,T];\mathbb{R}^n)$ is Banach when equipped with $\|\cdot\|_X = \|\cdot\|_{\infty,[0,T]}$

<u>Let</u> $\begin{cases} \dot{x} = f(t,x), t \in [0,T) \\ x(0) = x_0 \end{cases}$ ... (IVP)

Prop$^{\underline{n}}$. *$x \in C([0,T];\mathbb{R}^n)$ s.t. $x(t) = x_0 + \int_0^t f(s, x(s)).ds\ \forall t \in [0,T)$ (weak solution), $f \in C([0,T] \times \mathbb{R}^n)$ $\implies x \in C^1([0,T];\mathbb{R}^n)$ & solves (IVP).*

Th$^{\underline{m}}$. *<u>Picard's Theorem</u>: Suppose $f \in C([0,T] \times B_R(x_0);\mathbb{R}^n)$ where $T_1, R > 0$ & $|f(t,x) - f(t,y)| \le L|x - y| t \in [0,T],\ x, y \in B_R(x_0) \implies \exists T > 0$ & unique $x \in C([0,T];\mathbb{R}^n)$ s.t. $x(t) = x_0 + \int_0^t f(s, x(s)).ds\ \forall t \in [0,T)$ (weak sol$^{\underline{n}}$)*

<u>Let</u> $f \in C([0,T] \times \mathbb{R}^n) \dots$(GLL)

Lemma. *$x(t) = x_n(t),\ t \in [T_{n-1}, T_n)$ defined for $t \in [T_0, T_*)$ where $T_* = \lim_{n\to\infty} T_n \in (T_0, \infty) \implies x \in C^1([T_0, T_*); \mathbb{R}^n)$ & solves (IVP) on $[T_0, T_*)$.*

Def$^{\underline{n}}$. $x \in C^1([0, T_*); \mathbb{R}^n),\ T_* \in (0, \infty]$ is a <u>maximal solution</u> of (IVP) if $\forall y \in C^1([0, T); \mathbb{R}^n)$ solution of (IVP) $T \to T_*$ & $x = y$ on $[0, T_*$ then $T = T_*$.

Th$^{\underline{m}}$. .

1) *(GLL) $\implies \exists!$ maximal solution to (IVP).*
2) *Either $T_* = \infty$ or $|x(t)| \to \infty$ as $t \to T_*$*

Def$^{\underline{n}}$. T_* above is called a <u>maximal existance time</u> or if $T_* < \infty$ <u>blowup time</u>.

Note. F is conservative if $\exists E \in C^1(\mathbb{R}^n; \mathbb{R})$ *s.t.* $F(x) = -\nabla E(x)$. So $\ddot{x} = -\nabla E(x) \iff \begin{cases} \dot{x} = p \\ \dot{p} = -\nabla E(x) \end{cases}$ & $H(t) := \frac{1}{2}|\dot{x}| + E(x(t))$ is conserved ($\dot{H} \equiv 0$).

Prop$^{\underline{n}}$. *Suppose ∇E satisfies (GLL) & $\{y \in \mathbb{R}^n : E(y) \leq \varepsilon_0\}$ is bounded where $\varepsilon_0 := \frac{1}{2}|\dot{x}| + E(x_0)$ $\implies \begin{cases} \ddot{x} = -\nabla E(x) \\ x(0) = x_0,\ \dot{x}(0) = \dot{x}_0 \end{cases}$ has a unique solution*

Lemma. *<u>Gronwall 0</u>: $\xi \in C^1([0, T]; \mathbb{R}),\ \dot{\xi} \leq \alpha\xi + \beta,\ \alpha, \beta \in \mathbb{R} \implies \xi(t) \leq e^{\alpha t}\xi(0) + \beta\left(\frac{e^{\alpha t} - 1}{\alpha}\right)$*

Lemma. *<u>Gronwall 1</u>: $\alpha, \beta \in C^1([0, T]; \mathbb{R}),\ \xi \in C^1([0, T]; \mathbb{R}),\ \dot{\xi} \leq \alpha\xi + \beta,\ t \in (0, T)$ $\implies \xi(t) \leq e^{\int_0^t \alpha(r).dr}\xi(0) + \int_0^t e^{\int_s^t \alpha(r).dr}\beta(s).ds$*

Th$^{\underline{m}}$. *<u>Global existance for linear IVPs</u>: $A \in C^k([0, \infty); \mathbb{R}^{n\times n}),\ b \in C^k([0, \infty); \mathbb{R}^n),\ x_0 \in \mathbb{R}^n$ $\implies \exists! x \in C^1([0, \infty); \mathbb{R}^n)$ s.t. $\begin{cases} \ddot{x} = Ax + b,\ t > 0 \\ x(0) = x_0 \end{cases}$ & $x \in C^{k+1}([0, \infty); \mathbb{R}^n)$.*

Lemma. *<u>Generalised Gronwall</u>:*

a) $\alpha, \beta, \xi \in C([0, T]; \mathbb{R}),\ \alpha(t) > 0\ \forall t,\ \xi(t) \leq \beta(t) + \int_0^t \alpha(s)\xi(s).ds\ \forall t \in [0, t]$
$\implies \xi(t) \leq \beta(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \alpha(r).dr}.ds$

b) Assuming (a) & $\beta(t)$ monotonically increasing $\implies \xi(t) \leq \beta(t)e^{\int_0^t \alpha(s).ds}$

Th$^{\underline{m}}$. *Suppose $x, y \in C^1([0, T]; U)$*

- *f is Lipschitz: $|f(t, x) - f(t, x')| \leq L|x - x'|,\ t \in [0, T],\ x, x' \in U$*
- *$\varepsilon_M := \sup_{t\in[0,T]} |f(t, x) - g(t, x)| = \|f - g\|_{\infty,[0,T]\times U}$*

for (IVP)s $\begin{cases} \dot{x} = f(t, x) \\ x(0) = x_0 \end{cases}$ $\begin{cases} \dot{y} = g(t, y) \\ y(0) = y_0 \end{cases}$ $\implies |x(t) - y(t)| \leq e^{Lt}|x_0 - y_0| + te^{Lt}\varepsilon_M$

<u>Let</u> $\begin{cases} \dot{x}_\varepsilon = f(\varepsilon; t, x_\varepsilon) \\ x_\varepsilon(0) = z(\varepsilon) \end{cases}$ ...(IVP$_\varepsilon$) $\begin{cases} \dot{x}_0 = f(0; t, x_0) \\ . \end{cases}$ ...(IVP$_0$) $\begin{cases} \partial_\varepsilon f(0, t, x_0) + \partial_x f(0, t, x_0)x_1 \\ . \end{cases}$ ...(IVP$_1$)

Th$^{\underline{m}}$. *$f(\varepsilon, t, x) \in C^2([0, 1] \times [0, 1] \times \mathbb{R}^n; \mathbb{R}^n),\ z \in C^2([0, 1]; \mathbb{R}^n) \implies \exists T' \leq T,\ x_\varepsilon, x_0, x_1 \in C^1([0, T']; \mathbb{R}^n)$ solutions to (IVP$_\varepsilon$),(IVP$_0$),(IVP$_1$) & $\|x_\varepsilon - (x_0 - \varepsilon x_1)\|_{\infty,[0,T']} \leq C\varepsilon^2$ where C is independent of ε.*

Def$^{\underline{n}}$. <u>Forward Euler method</u>: for (IVP) $x_{n+1} := x_n + hf(nh, x_n)$

Lemma. *<u>Discrete Gronwall</u>: $\xi_n \in \mathbb{R},\ \alpha, \beta_n \in \mathbb{R},\ \alpha > 0,\ \xi_{n+1} \leq \alpha\xi_n + \beta_n$ $\implies \xi_n \leq \alpha^n\xi_0 + \sum_{i=0}^{n-1} \alpha^{n-1-i}\beta_i$*

Th$^{\underline{m}}$. *$f \in C^1([0, T]; \mathbb{R}^n)$, x is the solution to (IVP). Suppose $|x(t)|, |x_n| \leq R$, R independent of $h \in (0, 1]$ for $t, t_n \leq T \implies \exists C$ independent of $h \in (0, 1]$ s.t. $|x(t_n) - x_n| \leq Ch$ for $t_n \leq T$.*

Def$^{\underline{n}}$. <u>Implicit Euler method</u>: for (IVP) $x_{n+1} = x_n + hf(t_{n+1}, x_{n+1})$

Th$^{\underline{m}}$. *<u>Pēano's Theorem</u>: $f \in C([0,T'] \times B_R(x_0)) \implies \exists T>0,\ x \in C^1([0,T];\mathbb{R}^n)$ s.t.* $\begin{cases} \dot{x} = f(t,x) \\ x(0) = x_0 \end{cases}$

Th$^{\underline{m}}$. *<u>Arzela-Ascoli compactness theorem</u>: $x^h \in C([0,T];\mathbb{R}^n),\ \|x^h\|_{\infty,[0,T]} \le C$ (Uniform, Bounded), $|x^h(t)-x^h(s)| \le L|t-s|$ (Uniform, Lipschitz) $\implies \exists h_j \downarrow 0, x \in C([0,T];\mathbb{R}^n),\ \|x^{h_j}-x\|_{\infty,[0,T]} \to 0$ as $j \to \infty$*

<u>Let</u> $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ ...(A–IVP) $\qquad f:\mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz.

Def$^{\underline{n}}$. $x \in C^1((T_-,T_+);\mathbb{R}^n)$ solution to (A–IVP) is <u>maximal</u> if $\forall y \in C^1((T'_-,T'_+);\mathbb{R}^n)$ solution to (A–IVP), $T'_- \le T_-,\ T'_+ \ge T_+,\ y = x$ on $(T_-,T_+) \implies T'_- = T_-,\ T'_+ = T_+$.

Lemma. *Let x be a maximal solution to $\dot{x} = f(x)$*

1) *$t_0 \in \mathbb{R} \implies y(t) = x(t-t_0)$ is a maximal solution with existance interval $(T_- - t_0, T_+ - t_0)$*
2) *$z \in C^1((T'_-,T'_+);\mathbb{R}^n)$ is a maximal solution & $z(t_0) = x(t'_0) \implies T_{+/-} - t_0 = T'_{+/-} - t_0$ & $x(t-t_0) = z(t-t_0)\ \forall t$.*

Def$^{\underline{n}}$. $x \in C^1((T_-,T_+);\mathbb{R}^n)$ is a maximal solution then $\{x(t) : t \in (T_-,T_+)\} \subseteq \mathbb{R}^n$ is called the <u>orbit</u>. <u>{orbits}=phase portrait</u>.

Cor$^{\underline{ly}}$. .

a) *$\exists!$ maximal solution with $y(0) = y_0$*
b) *If $|T_{+/-}| < +\infty \implies |x(t)| \to \infty$ as $t \to T_{+/-}$*

Th$^{\underline{m}}$. *$\forall x_0 \in \mathbb{R}^n\ \exists!$ orbit passing through x_0. Let $x(t)$ be an associated maximal solution. Then one of three cases hold:*

1) *Equilibrium: $\dot{x}(t_0) = 0$ for some $t_0 \implies \dot{x} \equiv 0$*
2) *Periodic: $\dot{x}(t_0) \ne 0\ \forall t,\ \exists t_1 > 0$ s.t. $x(t_1) = x(0) \implies T_{+/-} = \pm\infty,\ x$ is periodic eith period t_1*
3) *Other: $x(t) \ne x(s)\ \forall t,s \in (T_-,T_+)$*

Lemma. *Let $x \in C^1((T_-,T_+);\mathbb{R}^n)$ be a maximal solution of $\dot{x} = f(x)$. If $x(t) \to x_+$ as $t \to T_+ \implies T_+ = \infty$ & $f(x_+) = 0$.*

Def$^{\underline{n}}$. <u>Flow map</u>: $\Phi(t,x_0) := x(t) \qquad t \in (T_-,T_+)$

1. $\Phi : \text{domain}(\Phi) \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$
2. $\Phi(0,x) = x$
3. $\Phi(s,\Phi(t,x)) = \Phi(t+s,x)$

If $G = \mathbb{Z} \implies$ <u>discrete dynamical system</u>.

Def$^{\underline{n}}$. $x_* \in \mathbb{R}^n$ is a <u>critical point</u> of f if $f(x_*) = 0$ (of <u>fixed point</u> of Φ or <u>equilibrium</u> of $\dot{x} = f(x)$). Call x_*:

i) <u>Stable</u> (in the sense of Lyapunov) if $\forall \delta > 0\ \exists \varepsilon > 0$ *s.t.* $x_0 \in B_\varepsilon(x_*) \implies \Phi(t,x_0) \in B_\delta(x_*)\ \forall t > 0$
ii) <u>Asymptotically stable</u> if $\exists \varepsilon > 0$ *s.t.* $x_0 \in B_\varepsilon(x_*) \implies \Phi(t,x_0) \to x_*$ as $t \to \infty$
iii) <u>Exponentially Stable</u> if $\exists \varepsilon, C, \alpha > 0$ *s.t.* $x_0 \in B_\varepsilon(x_*) \implies |\Phi(t,x_0) - x_*| \le Ce^{\alpha t}|x_0 - x_*|$
iv) <u>Unstable</u> if not stable.

Remark. (i)$\Leftarrow$(iii) $\Longleftrightarrow$ (ii)$\not\Rightarrow$(i)

Th$^{\underline{m}}$. *$f \in C^1(\mathbb{R}^n;\mathbb{R}^n)$, $f(x_*) = 0$, $\partial f(x_*) = A$ has eigenvalues Re$(\lambda_i) < 0 \implies x_*$ is exponentially stable.*

Lemma. *(Prop^n) $A \in \mathbb{R}^{n\times n}$, eigenvalues Re$(\lambda_i) < 0 \implies \exists C, \alpha > 0$ s.t $\|e^{At}\|_{\mathcal{F}} \leq Ce^{\alpha t}$*

Remark. Linearisation can only establish exponential stability, but nothing weaker.

Remark. Hartman-Grobman: x_* critical point of $f \in C^1$, $\text{Re}(\lambda_i) \neq 0 \lambda_i$ eigenvalues of $\partial f(x_*) \implies \exists$ homeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *s.t.* $\varphi(e^{At}u) = \Phi(t, \varphi(u))$ for $u, e^{At}u \in B_\varepsilon(x_*)$.

Th$^{\underline{m}}$. *Lyapunov's Theorem:*

- *$f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, x_* critical point of f.*
- *$L \in C^1(\mathbb{R}^n, \mathbb{R})$, $L(x_*) = 0, x_*$ is an isolated local minimiser of L.*

1) *If $\exists \delta_0 > 0$ s.t. $\nabla L \cdot f(x) \leq 0$ for $x \in B_{\delta_0}(x_*) \implies x_*$ is stable (Call L a Lyapunov function for x_*)*
2) *If $\exists \delta_0 > 0$ s.t. $\nabla L \cdot f(x) < 0$ for $x \in B_{\delta_0}(x_*)\backslash\{x_*\} \implies x_*$ is assymptotically stable (Call L a strict Lyapunov function for x_*)*
3) *If $\exists \delta_0 > 0$ s.t. $\nabla L \cdot f(x) > 0$ for $x \in B_{\delta_0}\backslash\{x_*\} \implies x_*$ is unstable.*

Def$^{\underline{n}}$. x_*-critical point of f (locally Lipschitz) then the basin of attraction of x_* is $\Sigma(x_*) := \{x_0 \in \mathbb{R}^n : \Phi(t, x_0) \to x_* \text{ as } t \to \infty\}$

Prop$^{\underline{n}}$. *Let L be a strict Lyapunov function for x_*-critical point of f (locally Lipschitz) $K_\mu \subseteq \{x \in \mathbb{R}^n : L(x) \leq \mu\}$ connected component containing x_* for $\mu > L(x_*)$. If K_μ is compact (closed & bounded) $\implies$ int$(K_\mu) \subseteq \Sigma(x_*)$.*

Cor$^{\underline{ly}}$. *Let x_* be an isolated local minimum of $E \implies (x_*, 0)$ is stable in the sense of Lyapunov.*

Cor$^{\underline{ly}}$. *Let $E \in C^2(\mathbb{R}^n;\mathbb{R})$, $\nabla E(x_*) = 0$, $\nabla^2 E(x_*)$ positive definite $\implies (x_*, 0)$ is an exponentially stable equilibrium of* $\begin{cases} \ddot{x} + c\dot{x} + \nabla E(x) = 0 \\ x(0) = x_0,\ \dot{x}(0) = p_0 \end{cases}$

Th$^{\underline{m}}$. *La Salle's Invariance Principle: $f \in C(\mathbb{R}^n;\mathbb{R}^n)$ locally Lipschitz $L \in C^1(\mathbb{R}^n;\mathbb{R})$*

i) *x_* is an isolated local minimum of L & an isolated critical point of L*
ii) *$\exists \delta > 0$ s.t. $\nabla L \cdot f(x) \leq 0$ for $x \in B_\delta(x_*)$*
iii) *If x is a maximal solution contained in $B_\delta(x_*)$ & $\nabla L(x(t)) \cdot f(x(t)) \equiv 0\ \forall t > 0 \implies x = x_*$
$\implies x_*$ is an asymtotically stable equilibrium.*

Cor$^{\underline{ly}}$. *x_* isolated local minimum & critical point of E, C positive definite $\implies (x_*, 0)$ is asymptotically stable.*