Linear Algebra

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Axioms for Number Systems

Let S be a number system $(S, +, \times)$
Axioms for Addition:

A1: $\forall \alpha, \beta \in S \quad \alpha + \beta = \beta + \alpha$

A2: $\forall \alpha, \beta, \gamma \in S \quad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

A3: $\exists 0 \in S \; s.t. \; \forall \alpha \in S \quad 0 + \alpha = \alpha + 0 = \alpha$

A4: $\forall \alpha \in S \; \exists (-\alpha) \in S \; s.t. \; \alpha + (-\alpha) = (-\alpha) + \alpha = 0$

Axioms for Multiplication:

M1: $\forall \alpha, \beta \in S \quad \alpha\beta = \beta\alpha$

M2: $\forall \alpha, \beta, \gamma \in S \quad (\alpha\beta)\gamma = \alpha(\beta\gamma)$

M3: $\exists 1 \in S \; s.t. \; \forall \alpha \in S \quad 1\alpha = \alpha 1 = \alpha$

M4: $\forall \alpha \in S^* \; \exists \alpha^{-1} \in S \; s.t. \; \alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$

Distributivity:

D1: $\forall \alpha, \beta, \gamma \in S \quad (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$

Def$^{\underline{n}}$. A set S with addition & multiplication satisfying A1-A4, M1-M4 & D1 is a field if $1 \neq 0$.

Vector Spaces

Def$^{\underline{n}}$. A vector space over a field K is a set V with addition & scalar multiplication, so $\forall \boldsymbol{v}, \boldsymbol{w} \in V \; \exists \boldsymbol{v} + \boldsymbol{w} \in V \; \& \; \forall \alpha \in K \; \forall \boldsymbol{v} \in V \; \exists \alpha\boldsymbol{v} \in V$

i) Addition Satisfies A1-A4
ii) $\alpha(\boldsymbol{v} + \boldsymbol{w}) = \alpha\boldsymbol{v} + \alpha\boldsymbol{w}$
iii) $(\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v}$
iv) $(\alpha\beta)\boldsymbol{v} = \alpha(\beta\boldsymbol{v})$
v) $1\boldsymbol{v} = \boldsymbol{v}1 \; \forall \boldsymbol{v} \in V$

$\Rightarrow$ Deductions:

i) $\alpha\mathbf{0} = \mathbf{0} \; \forall \alpha \in K$
ii) $0\boldsymbol{v} = \mathbf{0} \; \forall \boldsymbol{v} \in V$
iii) $-(\alpha\boldsymbol{v}) = (-\alpha)\boldsymbol{v} = \alpha(-\boldsymbol{v}) \; \forall \alpha \in K \; \forall \boldsymbol{v} \in V$
iv) If $\alpha\boldsymbol{v} = \mathbf{0}$ then $\alpha = 0$ or $\boldsymbol{v} = \mathbf{0}$

Linear Independence, Spanning & Bases of Vector Spaces

Def$^{\underline{n}}$. Let V be a vector space over K, $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in V$ the vectors are linearly independent if $\exists \alpha_1, \dots \alpha_n \in K$ not all 0 *s.t.* $\alpha_1 \boldsymbol{v}_1 + \dots + \alpha_n \boldsymbol{v}_n = \mathbf{0}$, otherwise linearly dependent.

Lemma. *If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly dependent, either $\boldsymbol{v}_1 = \mathbf{0}$ or $\boldsymbol{v}_r$ is a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{r-1}$.*

P$^{\underline{f}}$. Trivial QED

Def$^{\underline{n}}$. $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ span V if $\forall \boldsymbol{v} \in V$ $\exists \alpha_1, \dots, \alpha_n$ *s.t.* $\alpha_1 \boldsymbol{v}_1 + \dots + \alpha_n \boldsymbol{v}_n = \boldsymbol{v}$

Def$^{\underline{n}}$. If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ span V & are linearly independent they form a basis of V.

Def$^{\underline{n}}$. The unique scalars that determine any given $\boldsymbol{v} \in V$ are called the coordinates of $\boldsymbol{v}$.

Th$^{\underline{m}}$. *The Basis Theorem: Suppose $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ & $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$ are both bases of the vector space V, then $m = n$.*

Def$^{\underline{n}}$. The number of vectors n in a basis of the finite dimensional vector space V is called the dimension: $dim(V) = n$.

Sifting: Given $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r \in V$ successively look at $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r$ keep $\boldsymbol{v}_i$ unless $\boldsymbol{v}_i = \mathbf{0}$ or $\boldsymbol{v}_i$ is a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}$.

Lemma. *If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \boldsymbol{w}$ span V & $\boldsymbol{w}$ is a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ then $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ span V.*

P$^{\underline{f}}$. $\boldsymbol{w} = \alpha_1 \boldsymbol{v}_1 + \dots + \alpha_n \boldsymbol{v}_n$ now substitute $\boldsymbol{w}$ QED

Th$^{\underline{m}}$. *Suppose $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ span V then $\exists$ subsequence of vectors, a basis of V.*

P$^{\underline{f}}$. Sift $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ QED

Th$^{\underline{m}}$. *Suppose $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly independent in V. We can extend this to a basis of V.*

P$^{\underline{f}}$. Add $\boldsymbol{w}_1, \dots, \boldsymbol{w}_m$ & sift out $\boldsymbol{w}$'s. QED

Prop$^{\underline{n}}$. *The Exchange Lemma: Suppose $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ span V & $\boldsymbol{w}_1, \dots, \boldsymbol{w}_m$ are linearly independent in V then $m \leq n$.*

P$^{\underline{f}}$. Place $\boldsymbol{w}_1$ infront of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ & sift removing at lest one vector. now repeat for $\boldsymbol{w}_i$ removing at lesat one vector each time. Hence $m \leq n$. QED

Cor$^{\underline{ly}}$. *If n vectors form a basis of V then $n-1$ vectors cannot span V & $n+1$ vectors cannot be independent.*

P$^{\underline{f}}$. Of The Basis Theorem: Since $\boldsymbol{v}_i$'s span V & $\boldsymbol{w}_j$'s are linearly independent $n \leq m$ by Exchange Lemma. Since $\boldsymbol{w}_j$'s span V & $\boldsymbol{v}_i$'s are linearly independent $m \leq n$ by Exchange Lemma. Hence $n = m$. QED

Subspaces

Def$^{\underline{n}}$. A subspace of V is a non-empty subset $W \subset V$ *s.t.* W is closed under addition & scalar multiplication. i.e: $\boldsymbol{u}, \boldsymbol{v} \in W$ $\alpha, \beta \in K \implies \alpha \boldsymbol{u} + \beta \boldsymbol{v} \in W$

Prop$^{\underline{n}}$. *If W_1 & W_2 are subspaces of V then so is $W_1 \cap W_2$*

P$^{\underline{f}}$. Trivial QED

Note. $W_1 \cup W_2$ not necessarily a subspace.

Def$^{\underline{n}}$. Let W_1 & W_2 be subspaces of V then $W_1 + W_2$ is defined to be $\boldsymbol{v} \in V$ *s.t.* $\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2$ for some $\boldsymbol{w}_1 \in W_1$ & $\boldsymbol{w}_2 \in W_2$ or $W_1 + W_2 := \{\boldsymbol{w}_1 + \boldsymbol{w}_2 : \boldsymbol{w}_1 \in W_1 \ \boldsymbol{w}_2 \in W_2\}$

Prop$^{\underline{n}}$. *$W_1 + W_2$ is the smallest subspace to contain W_1 & W_2.*

P$^{\underline{f}}$. Any subspace of V containing W_1 & W_2 must contain $W_1 + W_2$ QED

At this point we drop bold face notation for vectors

Th$^{\underline{m}}$. *If V is a finite dimensional vector space & W_1, W_2 subspaces of V then:*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

P$^{\underline{f}}$. Let $\dim(W_1 + W_2) = r$ & $e_1, \dots, e_r$ be a basis of $W_1 \cap W_2$. Extend this to $e_1, \dots, e_r, f_1, \dots, f_s$ to be a basis of W_1 *s.t.* $\dim(W_1) = r + s$. Also extend to $e_1, \dots, e_r, g_1, \dots, g_t$ to be a basis of W_2 *s.t.* $\dim(W_2) = r + t$
$\forall w_1 \in W_1,\ w_1 = \alpha_1 e_1 + \cdots + \alpha_r e_r + \beta_1 f_1 + \cdots + \beta_s f_s$
$\forall w_2 \in W_2,\ w_2 = \gamma_1 e_1 + \cdots + \gamma_r e_r + \delta_1 g_1 + \cdots + \delta_t g_t$
hence $w_1 + w_2 = (\alpha_1 + \gamma_1)e_1 + \cdots + (\alpha_r + \gamma_r)e_r + \beta_1 f_1 + \cdots + \beta_s f_s + \delta_1 g_1 + \cdots + \delta_t g_t \in W_1 + W_2$ so e_i, f_j, g_k span $W_1 + W_2$
Suppose: $\alpha_1 e_1 + \cdots + \alpha_r e_r + \beta_1 f_1 + \cdots + \beta_s f_s + \gamma_1 g_1 + \cdots + \gamma_t g_t = 0$
then $\alpha_1 e_1 + \cdots + \alpha_r e_r + \beta_1 f_1 + \cdots + \beta_s f_s = -\gamma_1 g_1 - \cdots - \gamma_t g_t,$ *s.t.* $LHS \in W_1,\ RHS \in W_2$
$\implies$ both $\in W_1 \cap W_2$ with basis e_i.
Now $-\gamma_1 g_1 - \cdots - \gamma_t g_t = \delta_1 e_1 + \cdots + \delta_r e_r$ i.e: $\delta_1 e_1 + \cdots + \delta_r e_r + \gamma_1 g_1 + \cdots + \gamma_t g_t = 0$
e_i, g_k basis of $W_2 \implies$ all δ's & γ's $= 0$ leaving $\alpha_1 e_1 + \cdots + \alpha_r e_r + \beta_1 f_1 + \cdots + \beta_s f_s = 0$
e_i, f_j basis of $W_1 \implies$ all α's & β's $= 0$ so e_i, f_j, g_k are linearly independent, hence e_i, f_j, g_k are a basis of $W_1 + W_2$, hence

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

QED

Prop$^{\underline{n}}$. *$v_1, \dots, v_n \in V$ all linear combinations form a subspace of V.*

P$^{\underline{f}}$. Trivial QED

Def$^{\underline{n}}$. W_1, W_2 subspaces of V are complementary if $W_1 \cap W_2 = \{0\}$ & $W_1 \cup W_2 = V$.

Prop$^{\underline{n}}$. *W_1, W_2 subspaces of V, W_1 & W_2 are complementary $\iff$ $v \in V$ can be written uniquely as $v = w_1 + w_2$ where $w_1 \in W_1$ & $w_2 \in W_2$.*

P$^{\underline{f}}$. "$\implies$" Suppose W_1, W_2 complementary then $W_1 + W_2 = V$ so can find $w_1 \in W_1$ & $w_2 \in W_2$ *s.t.* $v = w_1 + w_2$ suppose $w_1' \in W_1$ & $w_2' \in W_2$ *s.t.* $v = w_1' + w_2'$ now $w_1 + w_2 = w_1' + w_2'$, $w_1 - w_1' = w_2' - w_2$ $LHS \in W_1$ $RHS \in W_2$ hence both $\in W_1 \cap W_2$ but $W_1 \cap W_2 = \{0\}$ hence $w_1 = w_1'$ & $w_2 = w_2'$.
"$\impliedby$" Suppose every $v \in V$ can be uniquely written $v = w_1 + w_2$, with $w_1 \in W_1$ $w_2 \in W_2$ (Obv.) $W_1 + W_2 = V$. If $0 \neq v \in W_1 \cap W_2$ then $v = v + 0,\ v \in W_1$ $v = 0 + v,\ v \in W_2$. Hence $W_1 \cap W_2 = \{0\} \implies W_1, W_2$ complementary. QED

Linear Transformations

Def$^{\underline{n}}$. Let U & V be vector spaces over K, a linear transformation or linear map T from U to V is a function $T : U \to V$ *s.t.* $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$ $\forall u_1, u_2 \in U$ & $\forall \alpha, \beta \in K$

Lemma. *i)* $T(0_U) = 0_V$

ii) $T(-u) = -T(u)$

P$^{\underline{f}}$. i) $T(0_U) = T(0_U + 0_U) = T(0_U) + T(0_U) \implies T(0_U) = 0_V$

ii) $T((-1)u) = (-1)T(u)$

QED

Prop$^{\underline{n}}$. *Let U, V be vector spaces over K, $u_1, \dots, u_n \in U$ basis $v_1, \dots, v_n \in V$ then $\exists! T$ s.t. $T : U \to V$ linear map with $T(u_i) = v_i$.*

P$^{\underline{f}}$. Let $u \in U$ then $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$ so $T(u) = T(\alpha_1 u_1 + \cdots + \alpha_n u_n) = \alpha_1 v_1 + \cdots + \alpha_n v_n = v \in V$ hence T is uniquely determined. QED

Def$^{\underline{\text{n}}}$. Let $T : U \to V$ be a linear map. The image of T $\underline{\text{Im}(T)}$ is the set of vectors $v \in V$ *s.t.* $v = T(u)$ for some $u \in U$. The kernel of T $\underline{\ker(T)}$ is the set of vectors $u \in U$ *s.t.* $T(u) = 0_V$.

$$\underline{\text{Im}(T)} := \{T(u) : u \in U\}, \quad \underline{\ker(T)} := \{u \in U : T(u) = 0_V\}$$

Prop$^{\underline{\text{n}}}$. *Let $T : U \to V$ be a linear map, then* $\text{Im}(T)$ *is a subspace of* V *&* $\ker(T)$ *is a subspace of* U.

P$^{\underline{\text{f}}}$. $\alpha v_1 + \beta v_2 = \alpha T(u_2) + \beta T(u_2) = T(\alpha u_1) + T(\beta u_2) = T(\alpha u_1 + \beta u_2) \in \text{Im}(T)$ for $u_1, u_2 \in U$ &$\alpha, \beta \in K$
$T(\alpha u_1 + \beta u_2) = T(\alpha u_1) + T(\beta u_2) = \alpha T(u_1) + \beta T(u_2) = \alpha 0_V + \beta 0_V = 0_V$ where $\alpha u_1 + \beta u_2 \in \ker(T)$
QED

Def$^{\underline{\text{n}}}$. Let $T : U \to V$ be a linear map $\dim(\text{Im}(T))$ is called the rank, $\dim(\ker(T))$ is called the nullity.

Th$^{\underline{\text{m}}}$. *Rank-Nullity Theorem: Let U, V be vector spaces over K with U finite dimensional. Let $T : U \to V$ be a linear map. Then*

$$\text{rank}(T) + \text{null}(T) = \dim(U)$$

P$^{\underline{\text{f}}}$. Since $\ker(T)$ is a subspace of U (Both finite dimensional). Let $\text{null}(T) = s$ & $e_1, \ldots, e_s$ be a basis of $\ker(T)$. Now extend to a basis of U: $e_1, \ldots, e_s, f_1, \ldots, f_r$. Now $\dim(U) = s+r$. $T(e_1), \ldots, T(e_s), T(f_1), \ldots, T(f_r)$ span $\text{Im}(T)$ & since $T(e_1), \ldots, T(e_s)$ all $= 0_V$ then $T(f_1), \ldots, T(f_r)$ span $\text{Im}(T)$.
Suppose $\alpha_1 T(f_1) + \cdots + \alpha_r T(f_r) = 0_V$ then $T(\alpha_1 f_1 + \cdots + \alpha_r f_r) = 0_V$ so $\alpha_1 f_1 + \cdots + \alpha_r f_r \in \ker(T)$ but $e_1, \ldots, e_s$ is a basis of $\ker(T)$ hence $\exists \beta_j \in K$ *s.t.* $\alpha_1 f_1 + \cdots + \alpha_r f_r = \beta_1 e_1 + \cdots + \beta_s e_s \implies \alpha_1 f_1 + \cdots + \alpha_r f_r - \beta_1 e_1 - \cdots - \beta_s e_s = 0_U$ but $e_1, \ldots, e_s, f_1, \ldots, f_r$ is a basis of U hence $\alpha_i, \beta_j = 0\ \forall i \implies f_1, \ldots, f_r$ linearly independent, hence $f_1, \ldots, f_r$ is a basis of $\text{Im}(T)$ hence $\text{rank}(T) + \text{null}(T) = r + s = \dim(U)$

QED

Cor$^{\underline{\text{ly}}}$. *Let $T : U \to V$ be a linear map where $\dim(U) = \dim(V) = n$. Then the following properties of T are equivalent:*

1. *T is surjective.*
2. $\text{rank}(T) = n$
3. $\text{null}(T) = 0$
4. *T is injective.*
5. *T is bijective.*

P$^{\underline{\text{f}}}$. (i) $\implies \text{Im}(T) = V \implies \text{rank}(T) = \dim(V) = n \implies$ (ii)
(ii) $\implies \text{Im}(T)$ subspace of V dimension $n \implies \text{Im}(T) = V \implies$ (i)
(ii) $\implies \dim(U) = n = \text{rank}(T) + \text{null}(T) \implies (T) = 0 \implies$ (iii)
(iii) $\implies \ker(T) = \{0_V\}$ $T(u_1) = T(u_2) \implies T(u_1 - u_2) = 0_V \implies u_1 - u_2 \in \ker(T) = \{0_V\} \implies u_1 = u_2 \implies$ (iv)
(iv) $\implies$ (iii)
finally (i)&(iv) $\iff$ (v) QED

Def$^{\underline{\text{n}}}$. If the above is satisfied then T is called non-singular, otherwise singular.

Def$^{\underline{\text{n}}}$. Addition & Scalar multiplication of linear maps: Let $T_1 : U \to V$ & $T_2 : U \to V$ then define $\alpha T_1 + \beta T_2 : U \to V$ to be $(\alpha T_1 + \beta T_2)(u) = \alpha T_1(u) + \beta T_2(u)\ \forall \alpha, \beta \in K\ \forall u \in U$

Def$^{\underline{\text{n}}}$. Composition of linear maps: Let $T_1 : U \to V$ & $T_2 : V \to W$ then define $T_2 \circ T_1 : U \to W$ to be $(T_2 \circ T_1)(u) = T_2(T_1(u))\ \forall u \in U$

Linear Transformations & Matrices

Let $T : U \to V$ be a linear map, where $\dim(U) = n$ & $\dim(V) =$, & $e_1, \dots, e_n$ is a basis of U & $f_1, \dots, f_m$ a basis V. Now

$$\begin{aligned} T(e_1) &= \alpha_{11} f_1 + \alpha_{21} f_2 + \cdots + \alpha_{m1} f_m \\ T(e_2) &= \alpha_{12} f_1 + \alpha_{22} f_2 + \cdots + \alpha_{m2} f_m \\ \vdots & \quad \vdots \\ T(e_n) &= \alpha_{1n} f_1 + \alpha_{2n} f_2 + \cdots + \alpha_{mn} f_m \vdots \\ & \text{with } \alpha_{ij} \in K \end{aligned} \tag{1}$$

or $T(e_j) = \sum_{i=1}^{m} \alpha_{ij} f_i$ for $1 \leq j \leq n$ & $A = (\alpha_{ij})$ is the matrix of the linear map T. This can be written $\left[T\right]_f^e = A$

Th$^{\underline{\text{m}}}$. *Let U, V be vector spaces over K of dimensions n, m respectively. For a giver choice of bases of U & V there is a one to one correspondence between the set $\mathrm{Hom}_K(U, V)$ of linear maps $U \to V$ & the set $K^{m \times n}$ of $m \times n$ matrices over K.*

P$^{\underline{f}}$. Use the above formulation QED

Prop$^{\underline{\text{n}}}$. *Let $T : U \to V$ be a linear map. Let $A = (\alpha_{ij})$ represent T wrt given bases of U & V. Then $T(u) = v \iff Au = v$ for $u \in U$ & $v \in V$.*

P$^{\underline{f}}$. $T(u) = T\left(\sum_{j=1}^{n} \lambda_j e_j\right) = \sum_{j=1}^{n} \lambda_j T(e_j) = \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} \alpha_{ij} f_i\right) = \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} \alpha_{ij} \lambda_j\right)}_{=*} f_i = \sum_{i=1}^{m} \underbrace{\mu_i}_{*=} f_i$

QED

Prop$^{\underline{\text{n}}}$. *Let $T_1, T_2 : U \to V$ be linear maps & A & B the respective matrices (& wrt the same bases). Then $\alpha T_1 + \beta T_2$ has matrix $\alpha A + \beta B$.*

P$^{\underline{f}}$. Trivial QED

Th$^{\underline{\text{m}}}$. *Let $T_1 : V \to W$ be a linear map with $\ell \times m$ matrix $A = (\alpha_{ij})$ & let $T_2 : U \to V$ be a linear map with $m \times n$ matrix $B = (\beta_{ij})$. Then the composite map $T_1 \circ T_2$ has matrix AB.*

P$^{\underline{f}}$. Similar to P$^{\underline{f}}$ of $T(u) = v \iff Au = v$ but $T_1(T_2(u)) = ABu$ QED

Elementary Operations & Rank of a Matrix

Elementary Row Operations:

(R1) For $i \neq j$ add a multiple of r_j to r_i (r_i, r_j are rows).

(R2) Interchange two rows.

(R3) Multiply a row by a non-zero scalar.

Def$^{\underline{\text{n}}}$. A matrix satisfying:

i) All zero rows below all non-zero rows.

ii) Let $r_1, \dots, r_s$ be the non-zero rows, then all r_i has a 1 as its first entry.

iii) The first non-zero entry of each row is strictly to the right og the first non-zero entry of the row above.

iv) If row i is non-zero all entries below the first non-zero element are zero.

is said to be in upper echelon form.

Def$^{\underline{n}}$. A matrix in upper echelon form satisfying:

v) If row i is non-zero then all entries above and below the first non-zero element are zero.

is said to be in row reduced form.

Th$^{\underline{m}}$. *Every matrix can be brought to row reduced form by elementary row operations.*

P$^{\underline{f}}$. Algorithm:

1) If α_{ij} & all entries below are zero move one place to the right $(i, j+1)$ & goto (1) unless $j = n$
2) If $\alpha_{ij} = 0$ but not all entries below are, apply (R2) to exchange rows.
3) If $\alpha_{ij} \neq 1$ apply (R3) using α_{ij}^{-1}.
4) Now apply (R1) *s.t.* all entries above & below that every entry are zero.
5) Move down one & right one $(i+1, j+1)$ unless $i = m$ or $j = n$. QED

Elementary Column Operations:

(C1) For $i \neq j$ adda multiple of c_j to c_i (c_i, c_j are columns).

(C2) Interchange two columns.

(C3) Multiply a column by a non-zero scalar.

Th$^{\underline{m}}$. *By applying elementary row & column operations a matrix can be brought into the form* $\left(\begin{array}{c|c} I_s & 0_{s,n-s} \\ \hline 0_{m-s,s} & 0_{m-s,n-s} \end{array}\right)$.

P$^{\underline{f}}$. Row reduce & use column operations. QED

Def$^{\underline{n}}$. A matrix in the above format is said to be in Smith normal form.

Lemma. *Let $T : U \to V$ be a linear map & $e_1, \ldots, e_n$ a basis of U then the rank of T is the largest independent subset of $T(e_1), dots, T(e_n)$.*

Def$^{\underline{n}}$. 1) The row space of A is the subspace of K^n spanned by the rows of A. The row rank is the dimension of the row space.

2) The column space of A is the subspace of K^n spanned by the columns of A. The column rank is the dimension of the column space.

Th$^{\underline{m}}$. *Suppose the linear map T has matrix A then* $\operatorname{rank}(T) = \operatorname{column\,rank}(A)$

P$^{\underline{f}}$. Use above lemma. QED

Th$^{\underline{m}}$. *Applying elementary row operations of elementary column operations does not change the row & column rank.*

P$^{\underline{f}}$. Obviousness. QED

Cor$^{\underline{ly}}$. *The number of non-zero rows in the Smith normal form of a matrix A is equal to both the row & column rank.*

P$^{\underline{f}}$. Elementary row & column operations don't change row or column ranks so:

$$\operatorname{row\,rank}(A) = \operatorname{row\,rank}(\text{Smith normal form of } A) = s = \operatorname{column\,rank}(\text{Smith normal form of } A) = \operatorname{column\,rank}(A)$$

QED

Cor$^{\underline{ly}}$. *The rank of a matrix A is equal to the number of rows that are non-zero in upper echelon form*

P$^{\underline{f}}$. Non-zero rows in upper echelon form are linearly independent. QED

Th$^{\underline{m}}$. *Let A be the augmented $n \times (m+1)$ matrix of a linear system. Let B be the matrix obtained from A by removing the last column. The system of equations have a solution* $\iff \operatorname{rank}(A) = \operatorname{rank}(B)$

The inverse of a Linear Transformation & of a Matrix

Def$^{\underline{n}}$. Let $T : U \to V$ be a linear transformation with corresponding matrix A $(m \times n)$. If $\exists T^{-1} : V \to U$ with $TT^{-1} = I_V$ & $T^{-1}T = I_U$ then T is said to be invertible & T^{-1} is called the inverse.
If so, A^{-1} is the $(n \times m)$ matrix & $AA^{-1} = I_m$ & $A^{-1}A = I_n$, then A is said to be invertible & A^{-1} is called the inverse.

Lemma. *Let A be a matrix of a linear map T. T is invertible $\iff$ A is invertible. T^{-1} & A^{-1} are unique.*

P$^{\underline{f}}$. Bijection between matrices & linear mpas. QED

Th$^{\underline{m}}$. *A linear map T is invertible $\iff$ T is non-singular. In particular if T is invertible then $m = n$ so only square matrices are invertible.*

P$^{\underline{f}}$. Map required to be injective so is inverse, hence bijection etc. QED

Prop$^{\underline{n}}$. *The row reduced form of an invertible $n \times n$ matrix A is I_n*

Elementary matrices:

- $\underline{E(n)^1_{\lambda,i,j}}$ $(i \neq j)$ $n \times n$ matrix like the identity butt with a non-zero entry λ in the $(i,j)^{th}$ position.
- $\underline{E(n)^2_{i,j}}$ Like the $n \times n$ identity with i^{th} & j^{th} rows interchanged.
- $\underline{E(n)^3_{\lambda,i}}$ $(\lambda \neq 0)$ like the $n \times n$ identity λ in the $(i,i)^{th}$ position.

Th$^{\underline{m}}$. *An invertible matrix is a product of elementary matrices.*

Th$^{\underline{m}}$. *Let A be an $n \times n$ matrix:*

i) *The homogeneous system $ax = 0$ has a non-trivial solution $\iff$ A is singular.*

ii) *The system $Ax = b$ has a unique solution $\iff$ A is non-singular.*

P$^{\underline{f}}$. i) The solution is nullspace(A) if T corresponds to A nullspace(T) $=$ ker(T) $= \{0\}$ $\iff$ null(T) $= 0$ $\iff$ T non-singular gives no solutions hence require A singular.

ii) If A singular then its nullity> 0 so nullspace(A) $\neq \{0\}$ $\implies$ no solutions OR solutions are $x+$nullspace(A) hence not unique.

QED

The Determinant of a Matrix

Def$^{\underline{n}}$. A permutation ϕ is said to be even if it is a composition of an even number of transpositions & $\underline{\text{sign}(\phi) = +1}$ & odd if a composition of an odd number of transpositions & $\underline{\text{sign}(\phi) = -1}$.

Def$^{\underline{n}}$. The determinant of an $n \times n$ matrix $A = (\alpha_{ij})$ is the scalar quantity

$$\underline{\det(A)} := \sum_{\phi \in S_n} \text{sign}(\phi)\alpha_{1\phi(1)} \cdots \alpha_{n\phi(n)}$$

Th$^{\underline{m}}$. *Elementary row operations affect the determinant as follows:*

i) $\det(I_n) = 1$.

ii) *Applying (R2) changes the determinants sign.*

iii) *If A has two equal rows* $\det(A) = 0$.

iv) *Applying (R1) does not change the determinant.*

v) *Applying (R3) multiplies the determinant by λ.*

Pf. i) $\alpha_{ij} = 0 \ \forall i \neq j$ so only identity permutation is non-zero $\det(I) = \alpha_{11} \cdots \alpha_{nn} = 1$.

ii) $\det(B) = \sum_{\phi \in S_n} \operatorname{sign}(\phi) \beta_{1\phi(1)} \cdots \beta_{n\phi(n)}$ let $\varphi = \phi \circ (i,j)$ now $\operatorname{sign}(\varphi) = -\operatorname{sign}(\phi)$ hence $= \sum_{\varphi \in S_n} -\operatorname{sign}(\phi) \alpha_{1\varphi(1)} \cdots \alpha_{n\varphi(n)} = -\det(A)$.

iii) Use part (ii) & swap rows that are the same now $\det(A) = -\det(A) \implies \det(A) = 0$.

iv)

$$\begin{aligned}
\det(B) &= \sum_{\phi \in S_n} \operatorname{sign}(\phi) \alpha_{1\phi(1)} \cdots (\alpha_{i\phi(i)} + \lambda \alpha_{j\phi(j)}) \cdots \alpha_{n\phi(n)} \\
&= \sum_{\phi \in S_n} \operatorname{sign}(\phi) \alpha_{1\phi(1)} \cdots \alpha_{n\phi(n)} + \lambda \sum_{\phi \in S_n} \operatorname{sign}(\phi) \alpha_{1\phi(1)} \cdots \alpha_{j\phi(j)} \alpha_{j\phi(j)} \cdots \alpha_{n\phi(n)}
\end{aligned}$$

Second term $= 0$ since $\alpha_{j\phi(j)}$ repeated is the same as two equal rows.

v) $\det(B) = \sum_{\phi \in S_n} \operatorname{sign}(\phi) \alpha_{1\phi(1)} \cdots \lambda \alpha_{i\phi(i)} \cdots \alpha_{n\phi(n)} = \lambda \sum_{\phi \in S_n} \operatorname{sign}(\phi) \alpha_{1\phi(1)} \cdots \alpha_{n\phi(n)} = \lambda \det(A)$

QED

Def$^{\underline{n}}$. A matrix is upper triangular if all entries below the leading diagonal are zero.

Cor$^{\underline{ly}}$. *The determinant of an upper triangular matrix is the product of its diagonal entries.*

Def$^{\underline{n}}$. Let $A = (\alpha_{ij})$ be an $m \times n$ matrix. Define the transpose A^T of A to be the $n \times m$ matrix $(\beta_{ij}) = (\alpha_{ji})$.

Th$^{\underline{m}}$. *Let $A = (\alpha_{ij})$ be an $n \times n$ matrix, then $\det(A^T) = \det(A)$.*

Pf.

$$\begin{aligned}
\det(A^T) &= \sum_{\phi \in S_n} \operatorname{sign}(\phi) \beta_{1\phi(1)} \cdots \beta_{n\phi(n)} \\
&= \sum_{\phi \in S_n} \operatorname{sign}(\phi) \alpha_{\phi(1)1} \cdots \alpha_{\phi(n)n} \\
&= \sum_{\phi \in S_n} \operatorname{sign}(\phi) \alpha_{1\phi(1)} \cdots \alpha_{n\phi(n)} \\
&= \det(A)
\end{aligned}$$

QED

Cor$^{\underline{ly}}$. *Elementary column operations affect the determinant as row operations do.*

Th$^{\underline{m}}$. *An $n \times n$ matrix A has $\det(A) = 0 \iff A$ is singular.*

Pf. Row reduce A to obtain A', $\det(A) = 0 \iff \det(A') = 0$, A' is upper triangular hence $\det(A) = 0 \iff$ one diagonal entry is $0 \iff A'$ has a zero row $\iff \operatorname{row rank}(A) < n \iff A$ is singular.

Lemma. *If E is an elementary matrix & B any matrix both $n \times n$ then $\det(EB) = \det(E)\det(B)$.*

Pf. $\det(E^1) = 1$, $\det(E^2) = -1$, $\det(E^3) = \lambda$ & consider row operations. QED

Th$^{\underline{m}}$. *For any two $n \times n$ matrices A & B: $\det(AB) = \det(A)\det(B)$.*

Pf. $\det(AB) = \det(E_1)\det(E_2 \cdots E_n B) = \det(E_1) \cdots \det(E_n)\det(B) = \det(E_1 \cdots E_n)\det(B)$
$= \det(A)\det(B)$ QED

Def$^{\underline{n}}$. Let $A = (a_{ij})$ be an $n \times n$ matrix. Let A_{ij} be the matrix $(n-1) \times (n-1)$ obtained by eliminating the i^{th} row & j^{th} column of A. $M_{ij} = \det(A_{ij})$ is called the $(i,j)^{th}$ minor of A.

Def$^{\underline{n}}$. $c_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \det(A_{ij})$ is called the $(i,j)^{th}$ cofactor of A.

Th$^{\underline{m}}$. *Let A be an $n \times n$ matrix.*

i) Expansion by i^{th} row: $\det(A) = \alpha_{i1}c_{i1} + \cdots + \alpha_{in}c_{in} = \sum_{j=1}^{n} \alpha_{ij}c_{ij}$.

ii) Expansion by j^{th} column: $\det(A) = \alpha_{1j}c_{1j} + \cdots + \alpha_{nj}c_{nj} = \sum_{i=1}^{n} \alpha_{ij}c_{ij}$.

P$\underline{^f}$.

$$\sum_{\substack{\phi \in S_n \\ \phi(n) = n}} \text{sign}(\phi)\alpha_{1\phi(1)} \cdots \alpha_{n\phi(n)} = \sum_{\phi \in S_{n-1}} \text{sign}(\phi)\alpha_{1\phi(1)} \cdots \alpha_{n-1\phi(n-1)}$$

now $\det(B) = \beta_{nn}N_{nn} = (-1)^{i+j}\alpha_{ij}M_{ij} = \alpha_{ij}c_{ij}$ hence $\det(A) = \sum \alpha_{ij}c_{ij}$ for fixed i or j. QED

Def$\underline{^n}$. Let A be an $n \times n$ matrix. Define the adjugate of A, adj(A), to be the $n \times n$ matrix with $(i,j)^{th}$ element the cofactor c_{ji}. i.e: the transpose of the matrix of cofactors.

Th$\underline{^m}$. $A adj(A) = \det(A)I_n = adj(A)A$

P$\underline{^f}$. $A = (\alpha_{ij})$ adj$(A) = (c_{ij})$ so Aadj$(A) = \left(\sum_{j=1}^{n} \alpha_{ij}c_{kj}\right)$. If $k \neq i$ $\sum_{j=1}^{n} \alpha_{ij}c_{kj} = 0$ hence Aadj$(A) = \det(A)I_n$ QED

Cor$\underline{^{ly}}$. *If* $\det(A) \neq 0$ *then* $A^{-1} = \frac{1}{\det(A)} adj(A)$.

Change of Basis & Equivalent Matrices

Prop$\underline{^n}$. *The change of basis matrix is invertible. i.e: if P is the change of basis matrix from e_i's to e'_i's & Q is the change of basis ,matrix from e'_i's to e_i's then $P = Q^{-1}$.*

P$\underline{^f}$. Consider $I_U : U \to U \to U$ where the first & last spaces use the same bases $\implies PQ = I_n \implies P = Q^{-1}$ QED

Prop$\underline{^n}$. *$Pv = v'$ where $v = \alpha_1 e_1 + \cdots + \alpha_n e_n$ & $v' = \beta_1 e'_1 + \cdots + \beta_n e'_n$.*

P$\underline{^f}$. Obviousness QED

Th$\underline{^m}$. *Let $T : U \to V$ be a linear map e, e' bases of U, f, f' bases of V*

$$P = [Id_U]^e_{e'} \quad Q = [Id_V]^f_{f'} \quad A = [T]^e_f \quad B = [T]^{e'}_{f'}$$

Then $B = QAP^{-1}$.

P$\underline{^f}$. $e \xrightarrow{P} e' \xrightarrow{B} f'$ hence BP $\quad e \xrightarrow{A} f \xrightarrow{Q} f'$ hence $QA \implies BP = QA$ QED

Def$\underline{^n}$. $A \& B$ $m \times n$ matrices are <u>equivalent</u> iff there are invertible matrices P $n \times n$ matrix & Q $m \times m$ matrix $s.t.$ $B = QAP$.

Th$\underline{^m}$. *Let A&B be $m \times n$ matrices over K the following conditions are equivalent:*

i) A&B are equivalent.

ii) A&B represent the same linear map wrt. different bases.

iii) A&B have the same rank.

iv) B can be obtained from A using elementary row & column operations.

P$\underline{^f}$. (i) $\iff$ (ii) by the previous Th$\underline{^m}$.
(ii) $\implies$ (iii) Since $A \& B$ represent the same linear map rank$(A) =$ rank$(T) =$ rank(B).
(iii) $\implies$ (iv) Both $A \& B$ have rank s so can be brought to Smith normal form by elementary row & column operations. i.e: $A \leftrightarrow S.N.F. \leftrightarrow B$
(iv) $\implies$ (i) $B = R_r R_{r-1} \cdots R_1 A C_1 C_2 \cdots C_s$ where R_i & C_i are row & column operation elementary matrices, hence $B = QAP$ QED

Similar Matrices, Eigenvectors & Eigenvalues

Def$^{\underline{n}}$. Two $n \times n$ matrices over K are said to be similar if $\exists$ an $n \times n$ matrix P which is invertible with $B = P^{-1}AP$.

Def$^{\underline{n}}$. A matrix similar to a diagonal matrix is said to be diagonalisable.

Def$^{\underline{n}}$. Let $T : V \to V$ be a linear map where V is a vector space over K. Suppose $v \in V$ non-zero & some $\lambda \in K$ we have $T(v) = \lambda v$. Then v is called an eigenvector of T & λ an eigenvalue of T corresponding to v.

Def$^{\underline{n}}$. Let A be an $n \times n$ matrix over K. Suppose some non-zero vector v & scalar $\lambda \in K$ satisfy $Av = \lambda v$. Then v is called an eigenvector of A & λ an eigenvalue of A corresponding to v.

Th$^{\underline{m}}$. *Let A be an $n \times n$ matrix. Then λ is an eigenvalue $\iff$ $\det(A - \lambda I_n) = 0$.*

P$^{\underline{f}}$. "$\implies$" $Av = \lambda v$ so $Av = \lambda I_n v$, $(A - \lambda I_n)v = 0$ for a solution $\det(A - \lambda I_n) = 0$.
"$\impliedby$" $\det(A - \lambda I_n) = 0$ then $A - \lambda I_n$ is singular, so has solutions $\implies \exists x$ *s.t.* $Av = \lambda v$. QED

Def$^{\underline{n}}$. For an $n \times n$ matrix A the equation $\det(A - \lambda x) = 0$ is called the characteristic equation of A & $\det(A - \lambda x)$ is called the characteristic polynomial.

Th$^{\underline{m}}$. *Similar matrices have the same characteristic equation & hence the same eigenvalues.*

P$^{\underline{f}}$. Let A&B be similar hence $B = P^{-1}AP$ for some P
$\det(B - xI_n) = \det(P^{-1}AP - xI_n) = \det(P^{-1}(A - xI_n)P) = \det(P^{-1})\det(A - xI_n)\det(P)$
$= \det(P^{-1})\det(P)\det(A - xI_n) = \det(A - xI_n)$ QED

Prop$^{\underline{n}}$. *Suppose A is upper triangular, the eigenvalues are just α_{ii}.*

P$^{\underline{f}}$. $\det(A - xI_n) = (\alpha_{11} - x) \cdots (\alpha_{nn} - x) = 0$ QED

Th$^{\underline{m}}$. *Let $T : V \to V$ be a linear map, the matrix of T is diagonal wrt. to some basis of V $\iff$ V has a basis consisting of eigenvectors of T.*
***Equivalently**: Let A be an $n \times n$ matrix over K. Then A is similar to a diagonal matrix $\iff$ K^n has a basis of eigenvectors of A.*

P$^{\underline{f}}$. "$\implies$" Suppose $A = (\alpha_{ij})$ is diagonal wrt. a basis of V hence $T(e_i) = \alpha_{ii}e_i$ so each e_i is an eigenvector.
"$\impliedby$" Suppose e_i's are a basis of V consisting of eigenvectors, so $T(e_i) = \lambda e_i$ for some $\lambda_i \in K$ so the matrix $A = (\alpha_{ij})$ with $\alpha_{ii} = \lambda_i$.

Th$^{\underline{m}}$. *Let $\lambda_1, \ldots, \lambda_r$ be distinct eigenvalues with $v_1, \ldots, v_r$ eigenvectors then $v_1, \ldots, v_r$ are linearly independent.*

P$^{\underline{f}}$. Induction on r. QED

Cor$^{\underline{ly}}$. *If A has distinct eigenvalues it is diagonalisable.*

Def$^{\underline{n}}$. The scalar product of two vectors $v = (\alpha_1, \ldots, \alpha_n)$ & $w = (\beta_1, \ldots, \beta_n)$ in $\mathbb{R}^n$ is defined to be $v \cdot w = \sum_{i=1}^{n} \alpha_i \beta_i$.

Def$^{\underline{n}}$. A basis $b_1, \ldots, b_n$ is called orthonormal if $b_i \cdot b_i = 1\ \forall i$ & $b_i \cdot b_j = 0\ \forall i \neq j$.

Def$^{\underline{n}}$. An $n \times n$ matrix A is said to be symmetric if $A^T = A$.

Def$^{\underline{n}}$. An $n \times n$ matrix A is said to be orthogonal if $A^T = A^{-1}$ or indeed $AA^T = A^TA = I_n$.

Prop$^{\underline{n}}$. *An $n \times n$ matrix A over $\mathbb{R}$ is orthogonal*
$\iff$ the rows $r_1, \ldots, r_n$ form an orthonormal basis of $\mathbb{R}^n$
$\iff$ the columns $c_1, \ldots, c_n$ form an orthonormal basis of $\mathbb{R}^n$.

P$^{\underline{f}}$. Orthogonal matrix A is invertible hence row & column ranks are n & hence the rows & columns form bases. $A^TA = I_n \implies r_i \cdot r_i = 1\ \forall i$ & $r_i \cdot r_j = 0\ \forall i \neq j$. Similarly for the converse. QED

Prop$^{\underline{n}}$. *Let A be a real symmetric matrix then A has a n eigenvalue in $\mathbb{R}$ & all complex eigenvalues of A are in $\mathbb{R}$.*

P$^{\underline{f}}$. $\det(A-\lambda I_n) \in \mathbb{R}[\lambda]_{deg=n}$ $\quad Av = \lambda v$ & $A\bar{v} = \bar{\lambda}\bar{v}$ (since $\bar{A} = A$). Also, $A^T = A$ so $v^T A^T = v^T A = \lambda v^T$

so $\left.\begin{array}{l}(v^T A)\bar{v} = \lambda v^T \bar{v} \\ v^T(A\bar{v}) = v^T \bar{\lambda}\bar{v}\end{array}\right\} \implies \lambda v^T \bar{v} = \bar{\lambda} v^T \bar{v} \implies (\lambda - \bar{\lambda}) v^T \bar{v} = 0$ since $v^T \bar{v} \neq 0 \implies \lambda = \bar{\lambda}$ hence $\lambda \in \mathbb{R}$

QED

Prop$^{\underline{n}}$. *Let A be a real symmetric matrix & let λ_1, λ_2 be two distinct eigenvalues v_1, v_2 then $v_1 \cdot v_2 = 0$.*

P$^{\underline{f}}$. $Av_1 = \lambda_1 v_1$ & $Av_2 = \lambda_2 v_2$ since $v_1^T A^T = \lambda_1 v_1^T$, $v_1^T A v_2 = \lambda_1 v_1^T v_2$
& similarly $v_2^T A v_1 = \lambda v_2^T v_1 = \lambda_2 v_2^T v_1 \implies v_1^T A v_2 = \lambda_2 v_1^T v_2$
$\therefore (\lambda_2 - \lambda_1) v_1^T v_2 = 0$ since $\lambda_2 - \lambda_1 \neq 0 \quad v_1 \cdot v_2 = 0$

QED

Th$^{\underline{m}}$. *Let A be a real symmetric matrix. $\exists$ a real orthogonal matrix P with $P^{-1}AP = P^T AP$ diagonal.*

P$^{\underline{f}}$. (Case of distinct eigenvalues only) $\lambda_i \in \mathbb{R}$ eigenvalues $\implies v_i \in \mathbb{R}^n$ eigenvectors $v_i \cdot v_j = v_i^T v_j = 0$ for $i \neq j$ since $v_i \neq 0$ $v_i \cdot v_i = \alpha_i > 0$ so replacing each v_i with $v_i/\sqrt{\alpha_i}$ we know $v_i \cdot v_i = 1$ $\forall i$. All the v_i are independent & form a basis (orthogonal). But $P^{-1}AP$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$.

QED