Geometry & Motion

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$\boldsymbol{r}(t) = (x(t), y(t), z(t))$ vector valued function of time $\boldsymbol{r} : I \to \mathbb{R}^3, I \subset \mathbb{R}$ s.t. $t \mapsto \boldsymbol{r}(t)$

Def$^{\underline{n}}$. $\mathcal{C} \subset \mathbb{R}$ is a <u>curve</u> (or <u>path</u>) if $\exists\ \boldsymbol{r} : I \to \mathbb{R}^n$ continuous s.t. $\mathcal{C} = \{\boldsymbol{r}(t) : t \in I\}$ were I is an interval.

Def$^{\underline{n}}$. The mapping $t \mapsto \boldsymbol{r}(t)$, $\boldsymbol{r} : I \to \mathbb{R}^n$ is called a <u>parameterisation</u> of $\mathcal{C}$ if it consists of n continuous functions $x_1(t), \ldots, x_n(t)$ the components of $\boldsymbol{r}$ of one variable t (not unique!)

If $\mathcal{C}$ is parameterised by $\boldsymbol{r}(t)$ & t is actually time them velocity is $\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}(t) = \lim_{h\to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$ component by component: $\boldsymbol{v}(t) = (\frac{dx_1}{dt}(t), \ldots, \frac{dx_n}{dt}(t))$.
Speed is the magnitude of velocity i.e: $\mathit{speed} = \|\boldsymbol{v}(t)\| = \left\|\frac{d\boldsymbol{r}}{dt}(t)\right\|$.
Acceleration: $\boldsymbol{a}(t) := \frac{d\boldsymbol{v}}{dt}(t)$

A closed curve or loop is where $\boldsymbol{r} : [a, b] \to \mathbb{R}^n$ & $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. To rule out intersections $\boldsymbol{r}(t_1) \neq \boldsymbol{r}(t_2)$ given $t_1 \neq t_2$ unless $t_1, t_2 \in \{a, b\}$.

Def$^{\underline{n}}$. A curve is <u>regular</u> if there exists a parameterisation s.t. $\frac{d\boldsymbol{r}}{dt}$ is defined and non-zero at all points so has no corners or cusps.

Def$^{\underline{n}}$. Let $\mathcal{C}$ be a curve parameterised by $\boldsymbol{r} : [a, b] \to \mathbb{R}^n$ the <u>length of $\mathcal{C}$:</u> $\underline{\ell(\mathcal{C}) := \int_a^b \left\|\frac{d\boldsymbol{r}}{dt}(t)\right\| .dt}$

Note. The length is independent of parameterisation.

Def$^{\underline{n}}$. Given a parameterisation of $\mathcal{C}$ we know $\boldsymbol{r}'(t)$ is <u>tangent</u> to $\mathcal{C}$ at $\boldsymbol{r}(t)$. $\underline{\boldsymbol{\tau}(t) := \frac{\boldsymbol{r}(t)}{\|\boldsymbol{r}'(t)\|}}$
<u>Principle normal</u> $\boldsymbol{n}$: $\underline{\boldsymbol{n}(t) := \frac{\boldsymbol{\tau}'(t)}{\|\boldsymbol{\tau}'(t)\|}}$
Curvature:
Define κ as <u>curvature</u>: $\underline{\kappa(t) := \frac{\|\boldsymbol{\tau}'(t)\|}{\|\boldsymbol{r}'(t)\|}}$
Define ρ as <u>radius of curvature</u>: $\underline{\rho(t) := \frac{1}{\kappa(t)}}$

Note. κ,ρ are scalars & independent of parameterisation.

The <u>binormal vector</u> $\boldsymbol{b}$: $\underline{\boldsymbol{b}(t) := \boldsymbol{\tau}(t) \times \boldsymbol{n}(t)}$

Note. $\boldsymbol{\tau}$,$\boldsymbol{n}$,$\boldsymbol{b}$ form the Frenet basis.

Def$^{\underline{n}}$. Let $f : \mathbb{R}^n \to \mathbb{R}$, then the <u>partial derivative</u> of f w.r.t. x_k is
$\frac{\partial f}{\partial x_k}(x_1, \ldots, x_n) := \lim_{h\to 0}\left[\frac{f(x_1,\ldots,x_k+h,\ldots,x_n) - f(x_1,\ldots,x_k,\ldots,x_n)}{h}\right]$

Def$^{\underline{n}}$. The <u>directional derivative</u> of $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\boldsymbol{x}$ in a direction $\boldsymbol{v}$, ($\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^n$) is $(D_{\boldsymbol{v}} f)(x) := \lim_{h\to 0} \frac{f(\boldsymbol{x}+h\boldsymbol{v}) - f(\boldsymbol{x})}{h}$

Useful approximation: $f(\boldsymbol{x} + \boldsymbol{h} \approx f(\boldsymbol{x}) + \boldsymbol{h} \cdot \nabla f(\boldsymbol{x})$
$\nabla f(\boldsymbol{x}) := \left(\frac{\partial f}{\partial x_1}(\boldsymbol{x}), \ldots, \frac{\partial f}{\partial x_n}(\boldsymbol{x})\right)$
So $\underline{(D_{\boldsymbol{v}} f)(\boldsymbol{x}) = \boldsymbol{v} \cdot \nabla f(\boldsymbol{x}) = \|\boldsymbol{v}\|\|\nabla f(\boldsymbol{x})\| \cos\theta}$
$\theta = 0$ gives max, $\theta = \pi$ gives min, $\theta = \pi \pm \frac{\pi}{2}$ gives level curve.

Def$^{\underline{n}}$. Tangent plane: $\pi(\boldsymbol{x}) : (\boldsymbol{r} - \boldsymbol{x}) \cdot \nabla f(\boldsymbol{x}) = 0$

Def$^{\underline{n}}$. The chain rule:
One dimension: $\frac{d}{dt} f(h(t)) = f'(h(t)) \cdot h'(t)$
Multivariable: $r : \mathbb{R} \to \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}$ let $g = f \circ \boldsymbol{r}$ then
$\frac{d}{dt} g(t) = \frac{d}{dt} f(\boldsymbol{r}(t)) = \nabla f(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$ or more generally: $\frac{df}{dt} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{dx_i}{dt}$

Def$^{\underline{n}}$. Area: $= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x_i = \int_a^b f(x).dx$
Volume:$= \lim_{n\to\infty} \sum_{i=1}^n f(\boldsymbol{x}_i)\Delta A_i = \iint_\Omega f.dA = \int_c^d \big(\int_a^b f(x,y).dx \big).dy$

$$V = \int_a^b \int_{g(x)}^{h(x)} f(x,y).dy.dx$$

where: $\Omega = \left\{ (x,y) : \begin{array}{c} a \le x \le b, \\ g(x) \le y \le h(x) \end{array} \right\}$

$$V = \int_c^d \int_{\xi(y)}^{\eta(y)} f(x,y).dx.dy$$

where: $\Omega = \left\{ (x,y) : \begin{array}{c} \xi(y) \le x \le \eta(y), \\ c \le y \le d \end{array} \right\}$

Area $= \iint_\Omega .dA$

$$\lim_{n\to\infty} \sum_{i=1}^n f(\boldsymbol{x}_i)\Delta V_i = \iiint_\Omega f.dV = \int_a^b \int_c^d \int_e^f f(x,y,z).dz.dy.dx$$

$$\iiint_\Omega f.dV = \int_a^b \int_{g(x)}^{h(x)} \int_{\xi(x,y)}^{\eta(x,y)} f(x,y,z).dz.dy.dx$$

where: $\Omega = \left\{ (x,y,z) : \begin{array}{c} a \le x \le b, \\ g(x) \le y \le h(x) \\ \xi(x,y) \le z \le \eta(x,y) \end{array} \right\}$

Def$^{\underline{n}}$. Polar Coordinates (r,θ): $\Delta A = \Delta r \Delta\theta\, r$ so $V = \int_\theta \int_r f(r,\theta) r.dr.d\theta$
Cylindrical Coordinates(r,θ,z): $\Delta V = \Delta r\Delta\theta\Delta z\, r$ so $V = \int_\theta\int_r\int_z f(r,\theta,z) r.dz.dr.d\theta$
Spherical Coordinates (ρ,θ,ϕ):

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$ so $\Delta V = \Delta\rho\Delta\phi\Delta\theta\rho^2\sin\phi$
$z = \rho\cos\theta$

so $\iiint_\Omega f.dV = \int_\phi\int_\theta\int_\rho f(\rho,\theta,\phi)\rho^2\sin\phi.d\rho.d\theta.d\phi$

Def$^{\underline{n}}$. Centre of Mass: Let $\bar{x} = \frac{1}{M}\iiint_\Omega x\rho.dV,\ \bar{y} = \frac{1}{M}\iiint_\Omega y\rho.dV,\ \bar{z} = \frac{1}{M}\iiint_\Omega z\rho.dV$ where $M = \iiint_\Omega \rho.dV$ for a body with density $\rho(x,y,z)$. Then $\boldsymbol{r}_{COM} := (\bar{x},\bar{y},\bar{z})$

Transformations & Generalised Coordinates:

2D Case: $\boldsymbol{\psi} : \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{\varphi} : \Gamma \to \Omega,\ (u,v) \mapsto (x(u,v), y(u,v))$ now
$\Delta A = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| \Delta u \Delta v$, so:

$$\iint_\Omega f.dA = \underbrace{\int_u\int_v}_{\Gamma} f(\boldsymbol{\psi}(u,v)) \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| .dv.du$$

Denote $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ as $\frac{\partial(x,y)}{\partial(u,v)}$ which is called the Jacobian Matrix

3D Case: $\boldsymbol{\psi} : \mathbb{R}^3 \to \mathbb{R}^3$, $\boldsymbol{\varphi} : \Gamma \to \Omega$, $(u,v,w) \mapsto (x(u,v,w), y(u,v,w), z(u,v,w))$ now $\Delta V = \left| \det \left(\frac{\partial(x,y,z)}{\partial(u,v,w)} \right) \right| \Delta u \Delta v \Delta w$, so:

$$\iiint_\Omega f.dV = \underbrace{\int_u \int_v \int_w}_{\Gamma} f(\boldsymbol{\psi}(u,v,w)) \left| \det \left(\frac{\partial(x,y,z)}{\partial(u,v,w)} \right) \right| .dw.dv.du$$

Line Integrals:

Line Integrals: $\ell(\mathcal{C}) = \int_a^b \|\boldsymbol{r}'(t)\|.dt = \int_{\mathcal{C}} .ds$ where $ds = \|\boldsymbol{r}'(t)\|.dt$ Let $f : \mathbb{R}^n \to \mathbb{R}$ & $\mathcal{C} \subset \mathbb{R}^n$ a curve, the line integral of f over $\mathcal{C}$ is $\int_{\mathcal{C}} f.ds = \int_a^b f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|.dt$

Arclength parameterisation (Natural Parameterisation): For $t \in [0,\ell]$ & $\ell(\mathcal{C}) = \int_0^s \|\boldsymbol{r}(t)\|.dt = s$ then $\forall s \in [0,\ell]$ $\frac{d}{ds}\int_0^s \|\boldsymbol{r}(t)\|.dt = \frac{d}{ds}s \implies \|\boldsymbol{r}'(s)\| = 1 \implies ds = dt$ (where ds is the arclength parameterisation) or $\int_0^s .dt = s$ so $\int_{\mathcal{C}} f.ds = \int_0^\ell f(\boldsymbol{r}(s)).ds$.

Vector Fields: $\boldsymbol{V} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{C} \subset \mathbb{R}^n$ a curve. The line integral of $\boldsymbol{V}$ over $\mathcal{C}$ is $\int_a^b \boldsymbol{v}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t).dt$ where $\boldsymbol{r}$ is a parameterisation of $\mathcal{C}$. The line integral differs only by the direction $\boldsymbol{r}$ traverses $\mathcal{C}$ i.e: by a factor of -1.

Gradient Fields: These are vector fields with the property that $\boldsymbol{v} = \nabla f$ for some $f : \mathbb{R}^n \to \mathbb{R}$

Th$^{\underline{m}}$. *Fundamental Theorem for Line Integrals of Vector Fields: Let $\mathcal{C}$ be a curve in $\mathbb{R}^n$, $\boldsymbol{r} : [a,b] \to \mathbb{R}^n$ with endpoints $\boldsymbol{a} = \boldsymbol{r}(a)$, $\boldsymbol{b} = \boldsymbol{r}(b)$ & let $\boldsymbol{v}$ be a gradient field i.e: $\boldsymbol{v} = \nabla f$. Then*

$$\int_{\mathcal{C}} \boldsymbol{v}.d\boldsymbol{\ell} = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

i.e: The line integral of a gradient field depends only upon the end points and not the path taken.

Cor$^{\underline{ly}}$.

$$\oint_{\mathcal{C}} \boldsymbol{v}.d\boldsymbol{\ell} = 0 \iff \boldsymbol{v} \text{ is a gradient field.}$$

Surface Integrals:

Curves: $\boldsymbol{r} : I \to \mathbb{R}^n$, $\quad I = [a,b]$

Surfaces: $\boldsymbol{r} : \Omega \to \mathbb{R}^n \quad \Omega = [a,b] \times [c,d]$

Tangent Plane: $\boldsymbol{p} : \mathbb{R}^2 \to \mathbb{R}^3$, $(h,k) \mapsto \mathbb{R}^3$ s.t.

$\boldsymbol{p}(h,k) = \boldsymbol{r}(u_0,v_0) + h\frac{\partial}{\partial u}\boldsymbol{r}(u_0,v_0) + k\frac{\partial}{\partial v}\boldsymbol{r}(u_0,v_0)$

Normal Vector: (to tangent plane) $\boldsymbol{n} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$ & $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|}$

Surface Integration: Area of $S = \iint_S .dS = \iint_\Omega \|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\|.du.dv$

For $f : \mathbb{R}^3 \to \mathbb{R}$, $\iint_S f.ds = \iint_\Omega f(\boldsymbol{r}(t))|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\|.du.dv$. For a surface S

Flux Integrals: $\iint \boldsymbol{v} \cdot \hat{\boldsymbol{n}}.ds$ where:

$$\hat{\boldsymbol{n}} = \frac{\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}}{\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\|}, \qquad ds = \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| .du.dv$$

hence $\iint_\Omega \boldsymbol{v} \cdot \hat{\boldsymbol{n}}.ds = \iint_\Omega \boldsymbol{v} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) .du.dv$

Taylor Series:

1D:

$$f(a+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n = f(a) + \cdots + \frac{f^{(n)}(a)}{n!} h^n + \cdots$$

2D:

$$\begin{aligned}
f(a+h, b+k) &= \sum_{n=0}^{\infty} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^n \frac{f(a,b)}{n!} \\
&= f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f(a,b) + \frac{1}{2!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f(a,b) + \cdots \\
&= f(a,b) + \frac{\partial f(a,b)}{\partial x} h + \frac{\partial f(a,b)}{\partial y} k \\
&\quad + \frac{1}{2!} \left[\frac{\partial f(a,b)}{\partial x^2} h^2 + 2\frac{\partial^2 f(a,b)}{\partial x \partial y} hk + \frac{\partial f(a,b)}{\partial y^2} k^2 \right] + \cdots
\end{aligned}$$

Critical Points

When $\nabla f = \mathbf{0}$:
Let $A = \frac{\partial f(a,b)}{\partial x^2}(\boldsymbol{x}_0)$, $B = \frac{\partial^2 f(a,b)}{\partial x \partial y}(\boldsymbol{x}_0)$, $C = \frac{\partial f(a,b)}{\partial y^2}(\boldsymbol{x}_0)$ & $\underline{D = AC - B^2}$
f is a maximum at $\boldsymbol{x}_0$ if $D > 0$ & $A < 0$
f is a minimum at $\boldsymbol{x}_0$ if $D > 0$ & $A > 0$
f is a saddle point at $\boldsymbol{x}_0$ if $D < 0$
If $D = 0$ at $\boldsymbol{x}_0$ no conclusion can be made about $f(\boldsymbol{x}_0)$