Differential Equations

Jack Betteridge

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Def$^{\underline{n}}$. A differential equation is a function of variables & their derivatives.

Def$^{\underline{n}}$. If we have a differential equation $F(t, x(t), \ldots, x^{(k)}(t)) = 0$ the order of the ordinary differential equation is the order of the highest derivative: k

Def$^{\underline{n}}$. If the independent variable does not appear explicitly in the ODE it is called autonomous.

Def$^{\underline{n}}$. Assuming and ODE can be written as $y^{(n)} = f(t, y, \ldots, y^{n-1)})$ then a solution is a function $\phi^{(n)}(t)$ s.t. $\phi^{(n)}(t) = f(t, \phi(t), \ldots, \phi^{(n)}(t))$ on some interval $\alpha \leq t \leq \beta \quad \alpha, \beta \in \mathbb{R} \cup \{-\infty, \infty\}$.

Def$^{\underline{n}}$. An n^{th} order ODE that can be written:

$$a_n(t)\frac{d^n y}{dt^n} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t)$$

is called linear homogeneous.

Def$^{\underline{n}}$. A function F satisfying $F'(t) = f(t)$ is called the anti-derivative of $f(t)$.

Th$^{\underline{m}}$. *The Fundamental Theorem of Calculus: Suppose $f : [a, b] \to \mathbb{R}$ continuous & define for $x \in [a, b]$ $G(x) = \int_a^x f(\tilde{x}).d\tilde{x}$ then $\frac{dG(x)}{dx} = f(x)$ furthermore $\int_a^b f(x).dx = F(b) - F(a)$ where F is the anti-derivative of f.*

Def$^{\underline{n}}$. Given and open interval I containing t_0 $x(t)$ is a solution of the initial value problem $\frac{dx(t)}{dt} = f(x, t)$ s.t. $x(t_0) = x_0$ on I is a continuous function $x(t)$ with $x(t_0) = x_0$ & $\frac{dx(t)}{dt} = f(x, t)$ $\forall t \in I$.

Th$^{\underline{m}}$. *Existence & Uniqueness: If $f(t, x)$ & $\frac{\partial f}{\partial x}(x, t)$ are continuous on $x \in (a, b)$ & $t \in (c, d)$ then $\forall x_0 \in (a, b)$ & $\forall t_0 \in (c, d)$ the initial value problem in th above def$^{\underline{n}}$ has a unique solution on some open interval $I \ni t$.*

First Order Differential Equations

Trivial Case: $\frac{dx}{dt} = f(t)$ so $x(t) = \int f(t).dt$
Linear Non-Homogeneous: $\frac{dx}{dt} + p(t)x = q(t)$, multiply each side by an integrating factor $e^{\int p(t).dt} = F(t)$ s.t. $\frac{d}{dt}[F(t)x(t)] = F(t)q(t)$ integrate to give: $x(t) = \frac{1}{F(t)}\int F(t)q(t).dt$
Separable Equations: $\frac{dx}{dt} = f(x)g(t)$ Check for constant solutions where $f(x) = 0$ otherwise $\int \frac{dx}{f(x)} = \int g(t).dt$
Autonomous ODEs: $\frac{dx}{dt} = f(t)$ Look for fixed points $f(x_*) = 0$ then $f'(x_*) < 0 \implies x_*$ is stable, $f'(x_*) > 0 \implies x_*$ is unstable.

Second Order Differential Equations

Constant Coefficients: $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$

- Homogeneous Case: $f(t) = 0$ solve the auxiliary equation: $a\lambda^2 + b\lambda + c = 0$:
  - if $b^2 - 4ac > 0$, 2 real roots k_1, k_2 then $x_c(t) = Ae^{k_1 t} + Be^{k_2 t}$
  - if $b^2 - 4ac = 0$, 1 repeated root k then $x_c(t) = Ae^{kt} + Bte^{kt}$
  - if $b^2 - 4ac < 0$, 2 complex roots $p \pm iq$ then $x_c(t) = e^{pt}(A\sin(qt) + B\cos(qt)) = Ee^{pt}\cos(qt - \phi)$

  These are called the complementary solutions.

- Non-homogeneous Case: Find the complementary solution & then depending on $f(t)$ try the following:

$\boldsymbol{f(t)}$	$\boldsymbol{x_p(t)}$
ae^{kt}	Ae^{kt}
ae^{kt} (k a root)	Ate^{kt} or At^2e^{kt}
$a\sin(\omega t)$ or $a\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
at^n	$P(t)$ polynomial of degree n
at^ne^{kn}	$P(t)e^{kt}$
$t^n(a\sin(\omega t) + b\cos(\omega t))$	$P_1(t)\sin(\omega t) + P_2(t)\cos(\omega t)$
$e^{kt}(a\sin(\omega t) + b\cos(\omega t))$	$e^{kt}(A\sin(\omega t) + B\cos(\omega t))$

The complete solution is of the form: $x(t) = x_c(t) + x_p(t)$.

Difference Equations

Euler's Method: Consider $\frac{dx}{dt} = f(x,t)$ s.t. $x(0) = x_0$ then $x(t+h) = x(t) + h\dot{x}(t) = x(t) + hf(t, x(t))$

$$\begin{aligned}
x_0 &= x(0) \\
x_1 &= x(h) = x(0+h) = x(0) + h\dot{x}(0) = x_0 + hf(0, x_0) \\
x_2 &= x(2h) = x(h+h) = x(h) + h\dot{x}(h) = x_1 + hf(h, x_1) \\
&\vdots \\
x_{k+1} &= x_k + hf(kh, x_k)
\end{aligned}$$

Def$^{\mathbf{n}}$. The order of a difference equation is the difference between the highest index of x and the lowest.

First order linear difference equation: $x_{n+1} = f(x_n, n)$ has the form $x_{n+1} = ax_n$ and has solution $x_{n+1} = a^n x_0$.

Second order linear difference equations: $ax_{n+2} + bx_{n+1} + cx_n = f(n)$

- Homogeneous: solve the auxiliary equation: $a\lambda^2 + b\lambda + c = 0$, for $f(n) = 0$
  - if $b^2 - 4ac > 0$, 2 real roots k_1, k_2 then $x_n = Ak_1^n + Bk_2^n$
  - if $b^2 - 4ac = 0$, 1 repeated root k then $x_n = Ak^n + Bnk^n$
  - if $b^2 - 4ac < 0$, 2 complex roots $p \pm ip = re^{\pm i\theta}$
- Non-homogeneous: Find the complementary solution and depending on $f(n)$ try:

$\boldsymbol{f(n)}$	$\boldsymbol{x_n}$
Polynomial in n degree$= d$	$P(x)$ polynomial of degree d
a^n	Ca^n, Cna^n, Cn^2a^n

The complete solution is of the form Homogeneous Solution + Non-homogeneous Solution
First order Autonomous Non-linear Difference Equations: A fixed point of the difference equation $x_{n+1} = f(x_n)$ is a point x_* s.t. $f(x_*) = x_*$. If $|f'(x_*)| < 1$ then x_* is stable & if $|f'(x_*)| > 1$ then x_* is unstable.

Systems of Linear First Order ODEs

Def$^{\underline{\text{n}}}$. A solution of the initial value problem $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}, t)$ s.t. $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ on an open interval $I \ni t_0$ is a continuous function $\boldsymbol{x} : I \to \mathbb{R}^n$ with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ & $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}, t) \quad \forall t \in I$.

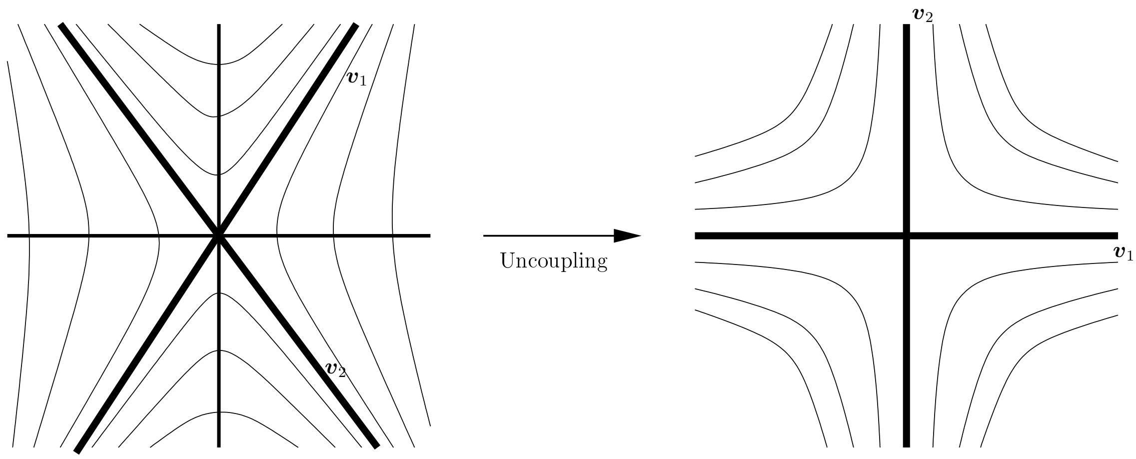
Def$^{\underline{\text{n}}}$. The Jacobian matrix of $f : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix of partial derivatives

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Th$^{\underline{\text{m}}}$. *Existence and Uniqueness: If $f(\boldsymbol{x}, t)$ & $Df(\boldsymbol{x}, t)$ are continuous for $\boldsymbol{x} \in U \subset \mathbb{R}^n,\ t \in (a, b)$ then $\forall \boldsymbol{x}_0 \in U$ & $\forall t \in (a, b)$ there exists a unique solution to $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}, t)$ on some open interval $\ni t$*

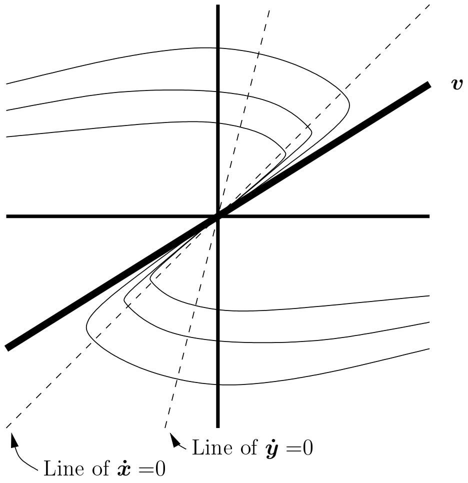
Coupled linear systems: Rearrange to form 2^{nd} order ODE & solve.
Homogeneous Linear systems 2×2 systems with Constant Coeffiecients:
Systems of the form $\begin{matrix} \dot{x} & = & px + qy \\ \dot{y} & = & rx + sy \end{matrix}$ if $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ & $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ then $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.
Solving $\det(A - \lambda I) = 0$ & finding eigenvalues & eigenvectors there are the following cases:

- Distinct real eigenvalues which leads to distinct eigenvectors $\lambda_1, \lambda_2, \boldsymbol{v}_1, \boldsymbol{v}_2$ has a solution $\boldsymbol{x}(t) = Ae^{\lambda_1 t}\boldsymbol{v}_1 + Be^{\lambda_2 t}\boldsymbol{v}_2$ with phase portrait:



  - if $\lambda_i > 0$ then arrows along $\boldsymbol{v}_i$ point outwards.
  - if $\lambda_i < 0$ then arrows along $\boldsymbol{v}_i$ point inwards.

- Repeated real eigenvalues which leads to one eigenvector $\lambda, \boldsymbol{v}$ has a solution $\boldsymbol{x}(t) = Be^{\lambda t}\boldsymbol{v} + Ce^{\lambda t}(\boldsymbol{a} + t\boldsymbol{v})$ (where $\boldsymbol{a}$ is found by solving $(A - \lambda I)\boldsymbol{a} = \boldsymbol{v}$) with phase portrait:



- if $\lambda > 0$ then arrows along $\boldsymbol{v}$ point outwards.
- if $\lambda < 0$ then arrows along $\boldsymbol{v}$ point inwards.

- Complex eigenvalues $\lambda_{\pm} = p \pm iq$ with eigenvectors $\boldsymbol{v}$ & $\bar{\boldsymbol{v}}$ or $\boldsymbol{v} = \boldsymbol{v}_1 \pm \boldsymbol{v}_2$ has the solutin $\boldsymbol{x}(t) = e^{pt}\left[(a\cos(qt) + b\sin(qt))\boldsymbol{v}_1 + (b\cos(qt) - a\sin(qt))\boldsymbol{v}_2\right]$ with phase portrait:

- if $p > 0$ then arrows point outwards.
- if $p < 0$ then arrows point inwards.
- if $q > 0$ spiral is clockwise.
- if $q < 0$ spiral is anticlockwise.