Analysis II

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Def$^{\underline{n}}$. Let $E \subset \mathbb{R}$, $x_0 \in \mathbb{R}$ x_0 is an accumulation point of E if $\forall \delta > 0$ $\exists x \in E$ $s.t.$ $|x - x_0| < \delta$.

Def$^{\underline{n}}$. Let $E \subset \mathbb{R}$, $f : E \to \mathbb{R}$, $x_0 \in E$. f is continuous at x_0 if $\forall \varepsilon > 0$ $\exists \delta > 0$ $s.t. \forall x \in E$ $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

Def$^{\underline{n}}$. Let $a \subset E$, f is continuous on A if it is continuous $\forall x_0 \in A$.

Def$^{\underline{n}}$. f is continuous if it is continuous on E.

Def$^{\underline{n}}$. $f : E \to \mathbb{R}$ is sequentially continuous at $c \in E$ if $\forall$ sequences $\{x_n\} \subset E$ $s.t. \lim_{n\to\infty} x_n = c \implies \lim_{n\to\infty} f(x_n) = f(c)$

Th$^{\underline{m}}$. *Continuity and sequential continuity are equivalent.*

P$^{\underline{f}}$. Suppose f continuous at c: $\forall \varepsilon > 0$ $\exists \delta > 0$ $s.t.$ $|x - c| < \delta$ $|f(x) - f(c)| < \varepsilon$. If $\{x_n\}$ is a sequence with $\lim_{x\to\infty} x_n = c$ then $\exists N \in \mathbb{N}$ $s.t.$ $\forall n > N$ $|x - c| < \delta$ so for $n > N$ $|f(x_n) - f(c)| < \varepsilon$. Suppose f not continuous at c: $\exists \varepsilon > 0$ $s.t.$ $\forall \delta > 0$ $\exists x \in E$ with $|x - c| < \delta$ but $|f(x_n) - f(c)| \geq \varepsilon$. This holds for $\delta \leq 1/n, \exists x_n \in E$ $s.t.$ $|x_n - c| < 1/n$, but $|f(x_n) - f(c)| \geq \varepsilon$. QED

Lemma. *Non-vanishing: Suppose $f : E \to \mathbb{R}$ is continuous at c. If $f(c) > 0$ $\exists \delta > 0$ $s.t.$ $f(x) > 0$ $\forall x \in E$ $s.t.$ $|x - c| < \delta$. Similarly, for $F(c) < 0$.*

P$^{\underline{f}}$. Suppose $f(c) > 0$. Take $\varepsilon = f(c)/2 > 0$ by def$^{\underline{n}}$ $\exists \delta > 0$ $s.t.$ $|f(x) - f(c)| < \varepsilon$ hence $f(x) > 0$ QED

Prop$^{\underline{n}}$. $c \in E \subset \mathbb{R}$ $f, g : E \to \mathbb{R}$. *Suppose f&g are continuous at c:*

i) $\alpha \cdot f$ is continuous at c $\forall \alpha \in \mathbb{R}$.

ii) $f + g$ is continuous at c.

iii) fg is continuous at c.

iv) Suppose $g(c) \neq 0$ $\exists \delta > 0$ $s.t.$ f/g is defined on $E \cap (c - \delta, c + \delta)$ & f/g is continuous at c.

P$^{\underline{f}}$. Convert each to sequential continuity. QED

Prop$^{\underline{n}}$. *Suppose $f : E \to \mathbb{R}$ continuous at c & $g : \mathrm{Range}(f) \to \mathbb{R}$ continuous at $f(c)$, then $g \circ f$ is continuous at c.*

P$^{\underline{f}}$. Convert to sequential continuity. QED

Th$^{\underline{m}}$. *Intermediate value: Let $f : [a, b] \to \mathbb{R}$ be continuous & $f(a) < f(b)$, then $\forall v \in (f(a), f(b))$ $\exists c \in (a, b)$ $s.t.$ $f(c) = v$.*

P$^{\underline{f}}$. Consider the set $A = \{x \in [a, b] : f(x) \leq v\}$, let $\sup A = c$ now $\exists \{x_n\} \subset A$ $s.t.$ $x_n \to c$. By continuity of f, $f(x_n) \to f(c) \leq v$. Supose $f(c) < v$ then non-vanishing lemma $\implies \exists \delta > 0$ $s.t.$ $|x - c| < \delta$ $f(x) < v$ which contradicts $\sup A = c$ since $c + \delta > c$ $\perp$, hence $f(c) = v$. QED

Def$^{\underline{n}}$. Let $f : (a, b) \backslash \{c\} \to \mathbb{R}$, $c \in (a, b)$, then we say f converges to ℓ as $x \to c$ i.e: $\lim_{x\to c} f(x) = \ell$ if $\forall x \in (a, b) \backslash \{c\}$ $\forall \varepsilon > 0$ $\exists \delta > 0$ $s.t.$ $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.

Lemma. *If ℓ_1 & ℓ_2 are limits of f at c then $\ell_1 = \ell_2$.*

$P^{\underline{f}}$. Suppose $\ell_1 \neq \ell_2$ WLOG $\ell_1 < \ell_2$ take $\varepsilon = \frac{\ell_2 - \ell_1}{2} > 0$. We know $|f(x) - \ell_1| < \varepsilon$ & $|f(x) - \ell_2| < \varepsilon$ $\forall x \in (a,b)\backslash\{c\}$ $\perp$ QED

Th$^{\underline{\mathbf{m}}}$. *$f : (a,b) \to \mathbb{R},\ c \in (a,b)$ then f continuous at c is equivalent to $\lim_{x\to c} f(x) = f(c)$.*

$P^{\underline{f}}$. They have the same definition! QED

Prop$^{\underline{\mathbf{n}}}$. *Sandwich Th$^{\underline{m}}$ & algebra of limits holds for continuous limits.*

$P^{\underline{f}}$. Convert to sequential continuity. QED

Def$^{\underline{\mathbf{n}}}$.

i) $f : (a,c) \to \mathbb{R}, \ell \in \mathbb{R}$ is a <u>left limit</u> of f at c if $\forall \varepsilon > 0\ \exists \delta > 0\ s.t.\ \forall x \in (c-\delta, c),\ |f(x) - \ell| < \varepsilon$. Denoted:

$$\underline{\lim_{x\to c^-} f(x) = \ell} \qquad \text{OR} \qquad \underline{\lim_{x\nearrow c} f(x) = \ell}$$

ii) $f : (c,b) \to \mathbb{R}, \ell \in \mathbb{R}$ is a <u>right limit</u> of f at c if $\forall \varepsilon > 0\ \exists \delta > 0\ s.t.\ \forall x \in (c, c+\delta)\ |f(x) - \ell| < \varepsilon$. Denoted:

$$\underline{\lim_{x\to c^+} f(x) = \ell} \qquad \text{OR} \qquad \underline{\lim_{x\searrow c} f(x) = \ell}$$

Def$^{\underline{\mathbf{n}}}$.

i) $f : (a,c] \to \mathbb{R}$ is <u>left continuous</u> at c if $\lim_{x\to c^-} f(x) = f(c)$

ii) $f : [c,b) \to \mathbb{R}$ is <u>right continuous</u> at c if $\lim_{x\to c^+} f(x) = f(c)$

Def$^{\underline{\mathbf{n}}}$. $f : (a,b)\backslash\{c\} \to \mathbb{R}$ is a <u>limit **to** $+\infty$</u> if $\forall M \in \mathbb{R}\ \exists \delta > 0\ s.t.\ \forall x \in (c-\delta, c+\delta)\ f(x) > M$ Denoted:

$$\lim_{x\to c} f(x) = +\infty$$

Similarly for $-\infty$.

Def$^{\underline{\mathbf{n}}}$. $f : (a, +\infty) \to \mathbb{R}$ is a <u>limit **at** $+\infty$</u> if $(\ell \in \mathbb{R} \cup \{-\infty, +\infty\})$, $\forall \varepsilon > 0\ \exists \delta > 0\ s.t.\ \forall x > k\ |f(x) - \ell| < \varepsilon$ Denoted:

$$\lim_{x\to +\infty} f(x) = \ell$$

Similarly for $f : (-\infty, b) \to \mathbb{R}$.

Th$^{\underline{\mathbf{m}}}$. *<u>Extreme value theorem</u>: Let $f : [a,b] \to \mathbb{R}$ be continuous, then it is bounded (above & below) & attains its maximum and minimum.*

$P^{\underline{f}}$.

i) Bounded Above: Suppose not $\forall M > 0\ \exists x \in [a,b]\ s.t.\ f(x) > M$ & $\forall n\ \exists x_n \in [a,b]\ s.t.\ f(x_n) > M$ (x_n) is bounded hence $\exists$ convergent subsequence $(x_{n_i}) \to x_0 \in [a,b]$ hence $\lim_{i\to\infty} f(x_{n_i}) = f(x_0)$ but since $f(x_n) > M$ $\lim_{i\to\infty} f(x_{n_i}) = +\infty$ $\perp$

ii) Attains Maximum: $A = \{f(x) : x \in [a,b]\} \neq \emptyset$ & bounded above $\implies$ $\sup A = M$ by def$^{\underline{n}}$ $\forall \varepsilon > 0$ $M - \varepsilon$ not an upper bound $\implies$ $M - \varepsilon \leq f(x) \leq M$ take $\varepsilon = 1/n$ & x_n $s.t.$ $M - 1/n \leq f(x_n) \leq M$ by sandwich Th$^{\underline{m}}$ $\lim_{n\to\infty} f(x_n) = M$, x_n has a convergent subsequence by Bolzano-Weierstrass.

iii) For bounded below & attains minimum define $g(x) = -f(x)$

QED

Def$^{\underline{\mathbf{n}}}$. $f : E \to \mathbb{R}$

i) f is <u>increasing</u> if $\forall x, y \in E\ s.t.\ x > y, f(x) > f(y)$

ii) f is <u>decreasing</u> if $\forall x, y \in E\ s.t.\ x > y, f(x) < f(y)$

iii) f is <u>non-decreasing</u> if $\forall x, y \in E\ s.t.\ x > y, f(x) \geq f(y)$

iv) f is <u>non-increasing</u> if $\forall x, y \in E\ s.t.\ x > y, f(x) \leq f(y)$

v) f is <u>monotone</u> if it is either increasing or decreasing.

Th$^{\underline{\mathbf{m}}}$. *If $f : [a,b] \to \mathbb{R}$ continuous & injective then f is monotone.*

$P^{\underline{f}}$. WLOG $f(a) < f(b)$

i) Want: $\forall c \in (a,b),\ f(c) \in (f(a), f(b))$
   $P^{\underline{f}}$. Suppose: $\exists c \in (a,b)$ $s.t.$ $f(c) > f(b)$ by intermediate value Th$^{\underline{m}}$ $\exists p \in (a,c)$ $s.t.$ $f(p) = f(b)$ but $p \neq b$ $\perp$ since f injective. QED

ii) Want: $\forall x_1, x_2 \in (a,b), x_1 < x_2$ then $f(x_1) < f(x_2) < f(b)$
   $P^{\underline{f}}$. Apply (i) to x_1 & restrict $f : [a,b] \to \mathbb{R}$ & apply (i) to x_2 on restricted f. Now $f(x_1) < f(x_2) < f(b)$ $\implies$ f is increasing.

QED

Th$^{\underline{m}}$. *Suppose $f : [a,b] \to \mathbb{R}$ increasing & surjective on [f(a),f(b)], then f is also continuous.*

$P^{\underline{f}}$. Take $c \in (a,b)$ & $\varepsilon > 0$. Let x_1 be $f(x_1) = \min\{f(b), f(c)+\varepsilon\}$ then $x_1 - c = \delta_1 > 0$. Let x_2 be $f(x_2) = \max\{f(a), f(c)-\varepsilon\}$ then $c - x_2 = \delta_2 > 0$. Take $\delta = min\{\delta_1, delta_2\}$ then $\forall x$ $s.t.$ $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$ & $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$ follows from f increasing & def$^{\underline{n}}$ of δ QED

Th$^{\underline{m}}$. *Inverse function theorem: Let $f : [a,b] \to \mathbb{R}$ be injective & continuous. $f^{-1} : \text{Range}(f) \to \mathbb{R}$ is well defined & continuous.*

$P^{\underline{f}}$. f must be monotone by previous Th$^{\underline{m}}$ WLOG f is increasing so Range$(f) = [f(a), f(b)]$ & $f^{-1} : [f(a), f(b)] \to [a,b]$ is surjective f^{-1} is increasing since f is increasing & by Th$^{\underline{m}}$ f^{-1} is continuous.QED

Def$^{\underline{n}}$. Let $f : (a,b) \to \mathbb{R}$ & $x \in (a,b)$, f is differentiable at x if $\exists \ell \in \mathbb{R}$ $s.t.$ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \ell$ & ℓ is called the derivative at x.

Th$^{\underline{m}}$. *Weierstrass-Caratheodory formulation of differentiation: Consider $f : (a,b) \to \mathbb{R}$ & $x_0 \in (a,b)$. f being differentiable at x_0 is equivalent to $\exists \phi : (a,b) \to \mathbb{R}$ $s.t.$ $f(x) = f(x_0) + \phi(x)(x - x_0)$ where ϕ is continuous at x_0 & furthermore $f'(x) = \phi(x)$.*

$P^{\underline{f}}$. "$\implies$" Suppose f differentiable at x_0. Set $\phi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{for } x \neq x_0 \\ f'(x_0) & \text{for } x = x_0 \end{cases}$ then $f(x) = f(x_0) + \phi(x)(x - x_0)$ since f differentiable at x_0.
"$\impliedby$" Assume $\exists \phi$ continuous at x_0 $s.t.$ $f(x) = f(x_0) + \phi(x)(x - x_0)$ then $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \phi(x) = \phi(x_0)$ by continuity, hence $f'(x) = \phi(x)$ QED

Cor$^{\underline{ly}}$. *If f is differentiable at x_0 it is continuous at x_0.*

Th$^{\underline{m}}$. *Algebra of derivatives: $f, g : (a,b) \to \mathbb{R}$ & $x_0 \in (a,b)$ f, g differentiable at x_0 then*

i) $f + g$ differentiable at x_0 & $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

ii) Product rule: fg differentiable at x_0 & $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

$P^{\underline{f}}$. By Weierstrass-Caratheodory: $f(x) = f(x_0) + \phi(x)(x - x_0)$, $g(x) = g(x_0) + \varphi(x)(x - x_0)$

i)

$$\begin{aligned}
(f+g)(x) &= f(x_0) + \phi(x)(x - x_0) + g(x_0) + \varphi(x)(x - x_0) \\
&= (f+g)(x_0) + [\phi(x) + \varphi(x)](x - x_0) \\
\text{hence } (f+g)'(x_0) &= (\phi + \varphi)(x_0) = f'(x_0) + g'(x_0)
\end{aligned}$$

ii)

$$\begin{aligned}
(fg)(x) &= \big(f(x_0) + \phi(x)(x - x_0)\big)\big(g(x_0) + \varphi(x)(x - x_0)\big) \\
&= (fg)(x_0) + [g(x_0)\phi(x) + f(x_0)\phi(x) + \phi(x_0)\varphi(x_0)(x - x_0)](x - x_0) \\
\text{Let } \theta(x) &= g(x_0)\phi(x) + f(x_0)\varphi(x) + \phi(x_0)\varphi(x_0)(x - x_0) \\
\text{hence } (fg)'(x_0) &= g(x_0)\phi(x_0) + f(x_0)\varphi(x_0) \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

QED

Th$^{\underline{m}}$. *Quotient rule:* $f, g : (a,b) \to \mathbb{R}$ *&* $x_0 \in (a,b)$ f, g *differentiable at* x_0. *Suppose* $g(x_0) \neq 0$ *then* f/g *is differentiable &* $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$

P$^{\underline{f}}$. Use Weierstrass-Caratheodory formulation again. QED

Th$^{\underline{m}}$. *Chain rule:* $f : (a,b) \to (c,d)$, $g : (c,d) \to \mathbb{R}$ $x_0 \in (a,b)$ f *differentiable at* x_0 *&* g *differentiable at* $f(x_0)$ *then* $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$

P$^{\underline{f}}$.

$$\begin{aligned} f(x) &= f(x_0) + \alpha(x)(x - x_0) \\ g(f(x)) &= g(f(x_0)) + \beta(f(x))[f(x) - f(x_0)] \\ &= g(f(x_0)) + \beta(f(x))\alpha(x)(x - x_0) \\ \text{hence } (g \circ f)'(x_0) &= \beta(f(x_0))\alpha(x_0) \\ &= g'(f(x_0))f'(x_0) \end{aligned}$$

QED

Th$^{\underline{m}}$. *Inverse function theorem 2.0: Let* $f : [a,b] \to [c,d]$ *continuous bijection* $x_\in(a,b)$ *&* f *differentiable at* x_0 *with* $f(x_0) \neq 0$ *then* f^{-1} *is differentiable at* $f(x_0)$ *&* $(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$

P$^{\underline{f}}$. Since f differentiable at x_0: $f(x) = f(x_0) + \phi(x)(x - x_0)$.
Let $y_0 = f(x_0)$ & $y = f(x)$ now $f(f^{-1}(y)) = f(f^{-1}(y_0)) + \phi(f^{-1}(y))[f^{-1}(y) - f^{-1}(y_0)]$
hence $y - y_0 = \phi(f^{-1}(y))[f^{-1}(y) - f^{-1}(y_0)]$
define $\theta(y) = \frac{1}{\phi(f^{-1}(y))}$ now $f^{-1}(y) = f^{-1}(y_0) + \theta(y)(y - y_0)$
hence f^{-1} differentiable & $(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$ QED

Def$^{\underline{n}}$.

i) $f : (a, x_0] \to \mathbb{R}$, f is <u>differentiable from the left</u> if $\exists \ell \in \mathbb{R}$ *s.t.* $\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \ell$ & ℓ is called the <u>left derivative</u>.

ii) $f : [x_0, b) \to \mathbb{R}$, f is <u>differentiable from the right</u> if $\exists \ell \in \mathbb{R}$ *s.t.* $\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \ell$ & ℓ is called the <u>right derivative</u>.

Th$^{\underline{m}}$. *A function* f *has a derivative at* x_0 *iff* $f'(x_0^+) = f'(x_0^-)$.

Def$^{\underline{n}}$.

i) f has a <u>local maximum at x_0</u> if $\forall x \in (x_0 - \delta, x_0 + \delta)$ $(\delta > 0)$ $f(x_0) \geq f(x)$

ii) f has a <u>strict local maximum at x_0</u> if $\forall x \in (x_0 - \delta, x_0 + \delta) \backslash \{x_0\}$ $(\delta > 0)$ $f(x_0) > f(x)$

iii) f has a <u>local minimum at x_0</u> if $\forall x \in (x_0 - \delta, x_0 + \delta)$ $(\delta > 0)$ $f(x_0) \leq f(x)$

iv) f has a <u>strict local minimum at x_0</u> if $\forall x \in (x_0 - \delta, x_0 + \delta) \backslash \{x_0\}$ $(\delta > 0)$ $f(x_0) < f(x)$

v) f has a <u>local extremum at x_0</u> if x_0 is a local maximum or minimum.

Lemma. $f : (a,b) \to \mathbb{R}$ $x_0 \in (a,b)$, x_0 *is a local extreme if* f *is differentiable t* x_0 *&* $f'(x_0) = 0$.

P$^{\underline{f}}$. WLOG x_0 is a maximum

i) take $h > 0$ small so $\frac{f(x_0 + h) - f(x_0)}{h} \geq 0$

ii) take $h > 0$ small so $\frac{f(x_0 - h) - f(x_0)}{-h} \geq 0$

Taking limits which correspond to left & right derivatives obtain $f'(x_0) = 0$ QED

Th$^{\underline{m}}$. *Rolle's Theorem: Let* $f : [a,b] \to \mathbb{R}$ *be continuous & differentiable on* (a,b) *with* $f(a) = f(b)$. *Then* $\exists x_0 \in (a,b)$ *s.t.* $f'(x_0) = 0$.

P$^{\underline{f}}$. Let f be maximised at $x_M \in (a,b)$ & minimised at $x_m \in (a,b)$ $x_M, x_m \notin \{a,b\}$ unless constant (trivial case!). Take $x_0 \in \{x_M, x_m\}$ *s.t.* $x_0 \notin \{a,b\}$ then $f'(x_0) = 0$ QED

Th$^{\underline{m}}$. *Mean Value Theorem (MVT): Let $f:[a,b]\to\mathbb{R}$ be continuous & differentiable on (a,b). Then $\exists x_0\in(a,b)$ s.t. $f'(x_0)=\frac{f(b)-f(a)}{b-a}$*

P$^{\underline{f}}$. Let $s=\frac{f(b)-f(a)}{b-a}$ define $g(x)=f(x)-s(x-a)$, g is continuous on $[a,b]$ & differentiable on (a,b) what's more $f(a)=g(a)$ & $g(b)=f(b)-s(b-a)=f(b)-(f(b)-f(a))f(a)$ by Rolle's Th$^{\underline{m}}$ $\exists x_0\in(a,b)$ s.t. $g'(x_0)=0$ hence $f'(x_0)=s$ QED

Cor$^{\underline{ly}}$. *$f:[a,b]\to\mathbb{R}$ continuous & differentiable on (a,b) & $\forall c\in(a,b)$ $f'(c)=0$ then f is constant.*

Def$^{\underline{n}}$. $f:[a,b]\to\mathbb{R}$ differentiable on (a,b) & one sided at a & b & f' continuous, then f is continuously differentiable. i.e: $f\in\mathcal{C}^1([a,b])$

Def$^{\underline{n}}$. $f:[a,b]\to\mathbb{R}$ is n times continuously differentiable at x_0 if $f^{(n-1)}$ is differentiable at x_0 & the n^{th} derivative id $f^{(n)}(x_0)=(f^{(n-1)})'(x_0)$.

Def$^{\underline{n}}$. $f:[a,b]\to\mathbb{R}$ if f is n times differentiable & $\exists f^{(n)}$ continuous then f is n times continuously differentiable.

Def$^{\underline{n}}$. $f:[a,b]\to\mathbb{R}$ is smooth if it is n times differentiable $\forall n\in\mathbb{N}$ i.e: $f\in\mathcal{C}^\infty([a,b])$

Def$^{\underline{n}}$. $f:[a,b]\to\mathbb{R}$ is convex if for $x_1,x_2\in[a,b]$ & for $t\in[0,1]$ we have $tf(x_1)+(1-t)f(x_2)\geq f(tx_1+(1-t)x_2)$ else f is concave.

Def$^{\underline{n}}$. An expression $\sum_{n=0}^\infty a_nx^n$ is called a power series around 0.
An expression $\sum_{n=0}^\infty a_n(x-x_0)^n$ is called a power series around x_0
Where $a_n\in\mathbb{R}$ $\forall n$ & x is a free variable.

Lemma. *Suppose $\sum a_nx^n$ converges for some x_0 & $c\in\mathbb{R}$ s.t. $|c|<|x_0|$ then the power series converges absolutely for c.*

P$^{\underline{f}}$. Since $\sum a_nx^n$ is convergent $\lim_{n\to\infty}|a_n||x_0|^n=0\implies$ bounded $\forall n$, $|a_nc^n|=|a_nx_0^n\left(\frac{c^n}{x_0^n}\right)|\leq Mr^n$ so by the comparison test $\sum a_nc^n$ converges QED

Th$^{\underline{m}}$. *Let $\sum a_nx^n$ be a power series & $S=\{x\in\mathbb{R}:\sum a_nx^n$ converges$\}$ either:*

i) $S=\emptyset$

ii) $S=\mathbb{R}$

iii) $\exists R\in\mathbb{R}$ *s.t.* $(-R,R)\subset S$ *(i.e: S may or maynot contain $\pm R$)*

Def$^{\underline{n}}$. If $\sum a_nx^n$ satisfies (i) we say the radius of convergence is 0.
If $\sum a_nx^n$ satisfies (ii) we say the radius of convergence is ∞.
If $\sum a_nx^n$ satisfies (iii) we say the radius of convergence is R.

Def$^{\underline{n}}$. Suppose (a_n) is a sequence & $\exists\ell\in\mathbb{R}\cup\{-\infty,+\infty\}$ *s.t.* $\lim_{n\to\infty}(\sup\{a_k:k\geq n\})=\ell$ then $\limsup_{n\to\infty}a_n=\ell$.

Th$^{\underline{m}}$. *Cauchy Root Test: $\sum_{n=1}^\infty z_n$ is:*

i) Convergent if $\limsup_{n\to\infty}|z_n|^{1/n}<1$

ii) Divergent if $\limsup_{n\to\infty}|z_n|^{1/n}>1$

P$^{\underline{f}}$. Let $a=\limsup_{n\to\infty}|z_n|^{1/n}$

1. $a<1$: $\exists\varepsilon>0$ *s.t.* $a+\varepsilon<1$ then $\exists N_0$ *s.t.* $\forall n>N_0$ $|z_n|^{1/n}<a+\varepsilon$ so $|z_n|<(a+\varepsilon)^n$ & $\sum_{n=1}^\infty(a+\varepsilon)^n$ is convergent by comparison hence $\sum_{n=0}^\infty|z_n|=\sum_{n=0}^{N_0}|z_n|+\sum_{n=N_0}^\infty|z_n|$ converges.

2. $a>1$: Choose *varepsilon* >0 *s.t.* $a-\varepsilon>1$ $\forall n$ $|z_n|^{1/n}>a-\varepsilon\implies|z_n|>(a-\varepsilon)^n>1\implies|z_n|\not\to0\implies\sum|z_n|$ diverges.

QED

Th$^{\underline{m}}$. *Hadamard's Test: Let $\sum a_n x^n$ be a power series*

i) If $\limsup_{n\to\infty} |a_n|^{1/n} = \infty$ then $\sum a_n x^n$ converges for $x = 0$ only.

ii) If $\limsup_{n\to\infty} |a_n|^{1/n} = 0$ then $\sum a_n x^n$ converges $\forall x \in \mathbb{R}$.

iii) If $\limsup_{n\to\infty} |a_n|^{1/n} = r$ $(0 < r < \infty)$ then $\sum a_n x^n$ has radius of convergence $1/r$.

P$^{\underline{f}}$. iii) If $|x| < R = 1/r$ $\quad \limsup_{n\to\infty} |a_n x^n|^{1/n} = r|x| < 1$ so by Cauchy's root test the power series converges.
If $|x| > R = 1/r$ $\quad \limsup_{n\to\infty} |a_n x^n|^{1/n} = r|x| > 1$ so by Cauchy's root test the power series diverges.

QED

Def$^{\underline{n}}$. If $\sum a_n x^n$ is a power series with radius of convergence R define $f : (-R, R) \to \mathbb{R}$ as $\underline{f(x) := \sum_{n=0}^{\infty} a_n x^n}$ & $\underline{f_N(x) := \sum_{n=0}^{N} a_n x^n}$

Prop$^{\underline{n}}$. *Let f be defined as above, then f is continuous & differentiable on $(-R, R)$.*

P$^{\underline{f}}$. Fix $x_0 \in (-R, R)$ & r *s.t.* $[x_0 - r, x_0 + r] \subset (-R, R)$ & $\rho = \max\{|x_0 - r|, |x_0 + r|\}$ $\forall \varepsilon > 0$ $\exists N$ *s.t.* $n > N$ $\sum_{n=1}^{\infty} |a_n|\rho^n$ satisfies $\sum_{n=N+1}^{\infty} |a_n|\rho^n < \varepsilon/3$. So $\sum_{n=N+1}^{\infty} |a_n||x^n| \le \sum_{n=N+1}^{\infty} |a_n|\rho^n < \varepsilon/3$. Hence

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x_0^n \right| \\ &\le \left| \sum_{n=0}^{N} a_n x^n - \sum_{n=0}^{N} a_n x_0^n \right| + \sum_{n=N+1}^{\infty} |a_n x^n| + \sum_{n=N+1}^{\infty} |a_n x_0^n| \\ &\le \sum_{n=0}^{\infty} a_n |x^n - x_0^n| \ + \ \frac{\varepsilon}{3} \ + \ \frac{\varepsilon}{3} \end{aligned}$$

Let $A = \sum_{n=0}^{N} |a_n|$. Since x^n continuous, $\forall n < N$ choose $\delta_n > 0$ *s.t.* $|x - x_0| < \delta_n \implies |x^n - x_0^n| < \frac{\varepsilon}{3A}$.
Let $\delta = \min\{\delta_1, \ldots, \delta_N, r\}$ *s.t.* if $|x - x_0| < \delta$ $\sum_{n=0}^{N} a_n |x^n - x_0^n| \le \sum_{n=0}^{N} a_n \left(\frac{\varepsilon}{3A}\right) \le \frac{\varepsilon}{3A} \sum_{n=0}^{N} a_n = \frac{\varepsilon}{3}$
hence for $|x - x_0| < \delta$ $|f(x) - f(x_0)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$
Differentiability: $f_N'(x) = \sum_{n=1}^{N} n a_n x^{n-1}$

QED

Lemma. *The power series $\sum a_n x^n$ & $\sum n a_n x^{n-1}$ have the same radius of convergence.*

P$^{\underline{f}}$. Observe $x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^n$ have the same radius of convergence. Hadamard's test gives: $\frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = \frac{1}{\limsup_{n\to\infty} |n a_n|^{1/n}}$ hence the radii of convergence of $\sum a_n x^n$ & $\sum n a_n x^{n-1}$ are the same.

QED

Th$^{\underline{m}}$. *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ & has radius of convergence R, then f is differentiable on $(-R, R)$ & $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$*

P$^{\underline{f}}$. Non-trivial.

QED

Def$^{\underline{n}}$.

$$\begin{aligned} \underline{\exp(x)} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2!} + \cdots \\ \underline{\sin(x)} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ \underline{\cos(x)} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &&= x - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \end{aligned}$$

Prop$^{\underline{n}}$. $\forall x, y \in \mathbb{R}$ $\exp(x + y) = \exp(x)\exp(y)$ & $\exp(-x) = 1/\exp(x)$

$P^{\underline{f}}$. Fix $y \in \mathbb{R}$. Define $f(x) = \exp(x+y)\exp(-x)$

$$\begin{aligned} f'(x) &= \left[\frac{d}{dx}\exp(x+y)\right]\exp(x) + \exp(x+y)\left[\frac{d}{dx}\exp(-x)\right] \\ &= \exp(x+y)\exp(-x) - \exp(x+y)\exp(-x) \\ &= 0 \end{aligned}$$

By MVT f is constant. Since $f(0) = \exp(y)$, $f(x) = \exp(y)\ \forall x$.
Take $y = 0$: $f(x) = \exp(x)\exp(-x)$, $f(0) = 1 \implies f(x) = 1\ \forall x \implies \exp(-x) = 1/\exp(x)$ QED

Prop$^{\underline{n}}$. exp *is the only solution to the ODE* $f'(x) = f(x)$ *with* $f(0) = 1$.

$P^{\underline{f}}$. Let f satisfy the ODE, set $g(x) = f(x)\exp(-x)$ now $g'(x) = f'(x)\exp(-x) - f(x)\exp(-x) = [f'(x) - f(x)]\exp(-x) = 0$ hence g is constant by MVT, so $f(x) = \alpha\exp(x) \implies f'(x) = \alpha\exp(x)$ $f'(0) = \alpha = f(0) = 1$ hence $\alpha = 1$ QED

Lemma. $f \in \mathcal{C}^n(a,b),\ x_0 \in (a,b)$ *The polynomial* $P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ *is the unique polynomial with the property that* $f^{(j)}(x_0) = p^{(j)}(x_0) \forall j \in \{0,\ldots,n\}$.

Def$^{\underline{n}}$. $f \in \mathcal{C}^\infty(a,b),\ x_0 \in (a,b)$ then $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is called the Taylor series of f around x_0.

Def$^{\underline{n}}$. A function $f \in \mathcal{C}^\infty(a,b)$ is analytic if $\forall x \in (a,b)$ the Taylor series around x_0 coincides with f on $(x_0 - \delta, x_0 + \delta)$ i.e: $f = P_n + R_n$ where R_n is small.

Th$^{\underline{m}}$. *Taylor's Theorem with remainder in Lagrange form: Suppose f is n times differentiable on $[x_0,x]$ & $n+1$ times differentiable on (x_0,x). Then:*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \ + \ R_n(x) \quad \textit{where} \quad R_n(x) = \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!}$$

for some $\xi \in (x_0, x)$.

$P^{\underline{f}}$. Let $g(y) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k$ so $g(x_0) = R_n(x),\ g(x) = 0$.
Let $h(y) = g(y) - A(x-y)^{n+1}$, $A = \frac{g(x_0}{(x-x_0)^{n+1}}$ so $h(x_0) = h(x) = 0$.
So by Rolle's Th$^{\underline{m}}$ $\exists c \in (x_0,x)$ *s.t.* $h'(c) = 0$
$g'(y) = \sum_{k=0}^{n} \frac{f^{(k)}(y)(x-y)^{k-1}}{(k-1)!} - \sum_{k=1}^{n} \frac{f^{(k+1)}(y)(x-y)^k}{k!} = -\frac{f^{(n+1)}(y)(b-y)^n}{n!}$
$\frac{d}{dy}A(b-y)^{(n+1)} = -A(n+1)(x-y)^n$ so $\frac{f^{(n+1)}(c)(x-c)^n}{n!} = \frac{g(x_0)}{(x-x_0)^{n+1}}(n+1)(x-c)^n$,
$g(x_0) = R_n(x_0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ QED

Cor$^{\underline{ly}}$. *Let* $f \in \mathcal{C}^\infty([a,b]),\ x_0 \in (a,b)$ *suppose the derivatives of all orders are uniformly bounded i.e:* $\exists k$ *s.t.* $\forall n \in \mathbb{N}\ \forall x \in [a,b]\ |f^{(n)}(x)| \le k$ *then* $\forall x, x_0 \in [a,b],\ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

$P^{\underline{f}}$. $\forall x \in [a,b]\ |R_n(x)| = \left|\frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)\right| \le K\frac{|x-x_0|^{n+1}}{(n+1)!}$ as $n \to \infty$ this tends to 0 hence
$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(n)}(x_0)(x-x_0)^k}{k!} = f(x)$ QED

Th$^{\underline{m}}$. *Cauchy's Mean Value Theorem: $f, g \in \mathcal{C}^1(a,b)$ & continuous on $[a,b]$ $\exists c \in (a,b)$ s.t. $g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$*

$P^{\underline{f}}$. Define $h(x) = g(x)[f(b) - f(a)] - f(x)[g(b) - g(a)]$, then $h(a) = h(b)$, h continuous on $[a,b]$ so $\exists c \in (a,b)$ *s.t.* $h'(c) = 0$ QED

Th$^{\underline{m}}$. *$f, g \in \mathcal{C}^{n+1}(a-\delta, a+\delta)$ If $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ & $g(a) = g'(a) = \cdots = g^{(n-1)}(a) = 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$.*

$P^{\underline{f}}$. $f(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ & $g(x) = \frac{g^{(n)}(a)}{n!}(x-a)^n + \frac{g^{(n+1)}(d)}{(n+1)!}(x-a)^{n+1}$ so

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^{(n)}(a)(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)}(x-a)^{n+1}}{g^{(n)}(a)n!(x-a)^n + \frac{g^{(n+1)}(d)}{(n+1)}(x-a)^{n+1}} = \lim_{x\to a}\frac{f^{(n)}(a)}{g^{(n)}(a)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

given $g^{(n)}(a) \neq 0$ when $f^{(n+1)}(a) = 0$ QED

Th$^{\underline{m}}$. *L'Hôpital's Rule: Let $x_0 \in (a,b)$ suppose f, g differentiable on (a,b) & $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ & $g'(x_0) \neq 0$ then $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$*

$P^{\underline{f}}$.

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}} = \frac{\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}} = \frac{f'(x_0)}{g'(x_0)}$$

QED

Th$^{\underline{m}}$. *$f, g : (a,b)\backslash\{x_0\} \to \mathbb{R}$ differentiable on $(a,b)\backslash\{x_0\}$ & $g(x) \neq 0$ $\forall x \in (a,b)\backslash\{x_0\}$*

i) If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ & $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$

ii) If $\lim_{x\to x_0} f(x) = \pm\infty$, $\lim_{x\to x_0} g(x) = \pm\infty$ & $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$

$P^{\underline{f}}$. For $f(x) = g(x) = 0$ & $\ell \in \mathbb{R}$, extend f, g on x_0: $f(x_0) = 0$, $g(x_0) = 0$ f, g continuous on $[x_0, x]$ & differentiable on (x_0, x). By Cauchy's MVT: $\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(c)}{g'(c)}$ for some $c \in (x_0, x)$ so

$$\lim_{x\to x_0^+}\frac{f(x)}{g(x)} = \lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x\to x_0^+}\frac{f'(c)}{g'(c)} = \ell$$

QED

Th$^{\underline{m}}$. *Suppose $f, g : (a,\infty) \to \mathbb{R}$ differentiable & $g'(x) \neq 0$ $\forall x \in (a,\infty)$*

i) If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ & $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \ell \in \mathbb{R}\cup\{-\infty,+\infty\}$ then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \ell$

ii) If $\lim_{x\to\infty} f(x) = \pm\infty$ $\lim_{x\to\infty} g(x) = \pm\infty$ & $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \ell \in \mathbb{R}\cup\{-\infty,+\infty\}$ then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \ell$

$P^{\underline{f}}$. Let $t = 1/x$, $F(t) = f(1/t)$, $G(t) = g(1/t)$ now $\lim_{x\to\infty} f(x) = \lim_{t\to 0} f(1/t) = \lim_{t\to 0} F(t)$ & $\lim_{x\to\infty} g(x) = \lim_{t\to 0} g(1/t) = \lim_{t\to 0} G(t)$ now $\frac{F'(t)}{G'(t)} = \frac{f'(1/t)(1/t^2)}{g'(1/t)(1/t^2)} = \frac{f'(x)}{g'(x)}$ which reduces to previous th$^{\underline{m}}$ QED