Analysis I

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15/2/13

Workbook 1

<u>Power Rule</u>: For $x, y \in \mathbb{R}$ & $x, y > 0$ If $x < y$ $\forall n \in \mathbb{N} \iff x^n < y^n$ This fails for x or $y < 0$.
Proof. Inductive.

Def$^{\underline{n}}$. $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

<u>Modulus</u>: $\big||x|\big| = |x|, \quad |xy| = |x||y|, \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
<u>Interval Property</u>: For $x, y \in \mathbb{R}$ & $b > 0$ $|x| < b \iff -b < x < b$ if $x = y - a$ $|y - a| < b \iff a - b < y < a + b$
<u>Triangle Inequality</u>: For $x, y \in \mathbb{R}$ $|x + y| \leq |x| + |y|$ *Proof.* Square both sides.

Def$^{\underline{n}}$. <u>Arithmetic Mean</u> $:= \frac{a_1 + \ldots a_n}{n}$, <u>Geometric Mean</u> $:= \sqrt[n]{a_1 \cdots a_n}$

Workbook 2

Def$^{\underline{n}}$.

$$\begin{aligned}
(a_n) \text{ is strictly increasing} &\iff \forall n,\ a_n < a_{n+1} \\
(a_n) \text{ is increasing} &\iff \forall n,\ a_n < a_{n+1} \\
(a_n) \text{ is strictly decreasing} &\iff \forall n,\ a_n < a_{n+1} \\
(a_n) \text{ is decreasing} &\iff \forall n,\ a_n < a_{n+1} \\
(a_n) \text{ is monotonic} &\iff (a_n) \text{ is increasing or } (a_n) \text{ is decreasing (or both)} \\
(a_n) \text{ non-monotonic} &\iff a_n \text{ is neither increasing nor decreasing} \\
(a_n) \text{ is bounded above} &\iff \exists U \in \mathbb{R}\ s.t. \forall n, a_n \leq U \ (U \text{ is upper bound}) \\
(a_n) \text{ is bounded below} &\iff \exists L \in \mathbb{R}\ s.t. \forall n, a_n \geq L \ (L \text{ is upper bound}) \\
(a_n) \text{ is bounded} &\iff (a_n) \text{ is bounded above and below}
\end{aligned}$$

Def$^{\underline{n}}$. $(a_n) \to \infty \iff \forall C > 0\ \exists N \in \mathbb{N}\ s.t.\ \forall n > N,\ a_n > C$

Th$^{\underline{m}}$. (a_n) & (b_n)*-sequences & $b_n > a_n$ $\forall n$, if $(a_n) \to \infty$ then $(b_n) \to \infty$*

Th$^{\underline{m}}$. *If $(a_n) \to \infty$ & $(b_n) \to \infty$ then $(a_n + b_n) \to \infty,\ (a_n b_n) \to \infty,\ (ca_n) \to \infty$ for $c > 0$ & $(ca_n) \to -\infty$ for $c < 0$*

Def$^{\underline{n}}$. $(a_n) \to 0 \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ s.t.\ \forall n > N,\ |a_n| < \varepsilon$

Lemma. $(a_n) \to \infty \implies \left(\frac{1}{a_n}\right) \to 0$ *("$\Longleftarrow$" is false)*

<u>Absolute Value Rule</u>: $(a_n) \to 0 \iff (|a_n|) \to 0$
<u>Sandwich Rule</u>: (For null sequences) If $(a_n) \to 0$ & $0 \leq |b_n| \leq a_n$ $\forall n \implies (b_n) \to 0$
<u>Sum Rule</u>: (For null sequences) If $(a_n) \to 0 \& (b_n) \to 0$ then $\forall c, d \in \mathbb{R}$ $(ca_n + db_n) \to 0$
<u>Product Rule</u>: (For null sequences) If $(a_n) \to 0 \& (b_n) \to 0$ then $(a_n b_n) \to 0$

Workbook 3

Def$^{\underline{n}}$. $(a_n) \to a \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ s.t.\ \forall n > N,\ |a_n - a| < \varepsilon$

Lemma. $(a_n) \to a \iff (a_n - a) \to 0$

Uniqueness of Limits: A sequence cannot converge to more than one limit.

Th$^{\underline{m}}$. *Every convergent sequence is bounded.*

Sum Rule: If $(a_n) \to a$ & $(b_n) \to b$ then $\forall c, d \in \mathbb{R}, (ca_n + db_n) \to ca + db$
Product Rule: If $(a_n) \to a$ & $(b_n) \to b$ then $(a_n b_n) \to ab$
Quotient Rule: If $(a_n) \to a$ & $(b_n) \to b,\ b, b_n \neq 0\ \forall n$ then $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$

P$^{\underline{f}}$. (Of Quotient Rule) $(b_n) \to b \implies (bb_n) \to b^2 \implies bb_n > \frac{b^2}{2}$ for some n now: $(bb_n - b^2) \to 0$ divide by $b^2 b_n$ gives $\left(\frac{1}{b} - \frac{1}{b_n}\right) \to 0$ or $\left|\frac{1}{b} - \frac{1}{b_n}\right| = \left|\frac{b_n - b}{bb_n}\right|$ since $\frac{b^2}{2} < bb_n,\ \frac{2}{b^2} > \frac{1}{bb_n} \implies \left|\frac{b - b_n}{bb_n}\right| \leq \frac{2}{b^2}|b - b_n|$ since $(b_n) \to b,\ \frac{2}{b^2}|b - b_n| \to 0 \implies \left(\frac{1}{b_n}\right) \to \frac{1}{b}$ (again given $b \neq 0$). QED

Sandwich Rule: If $(a_n) \to \ell$ & $(b_n) \to \ell$ & $a_n \leq c_n \leq b_n$ (eventually – see shift rule)then $(c_n) \to \ell$

Def$^{\underline{n}}$. (a_n) satisfies a property eventually if $\exists N \in \mathbb{N}\ s.t.\ (a_{N+n})$ satisfies the property.

Lemma. *If a sequence is eventually bounded, it is bounded.*

Shift Rule: $(a_n) \to a \iff (a_{N+n}) \to a$

Lemma. *If $(a_n) \to a$ & $a_n \geq 0\ \forall n$ then $a \geq 0$ (Strict inequality is false)*

Th$^{\underline{m}}$. *If $(a_n) \to a$ & $(b_n) \to b$ & eventually $a_n \leq b_n$ then $a \leq b$*

Cor$^{\underline{ly}}$. *Closed interval rule: If $(a_n) \to a$ & eventually $A \leq a_n \leq B$ then $A \leq a \leq B$*

Def$^{\underline{n}}$. A subsequence of (a_n) is (a_{n_i}) where (n_i) is a strictly increasing sequaence in $\mathbb{N}$.

Lemma. *Every subsequence of a bounded sequence is bounded.*

Th$^{\underline{m}}$. *Every sequence has a monotonic subsequence.*

Workbook 4

Bernoulli's inequality: For $x > -1$ & $n \in \mathbb{N} : (1 + x)^n \geq 1 + nx$.

Prop$^{\underline{n}}$. *If $x > 0$ then $(x^{1/n}) \to 1$.*

Lemma. *Ratio Lemma: Let $a_0, a_1, \ldots$ be a sequence where $a_n > 0 \forall n$. If $\frac{a_{n+1}}{a_n} \leq \ell$ eventually and $0 < l < 1$ then $(a_n) \to 0$.*

Cor$^{\underline{ly}}$. *Let $a_0, a_1, \ldots$ be a sequence where $a_n > 0 \forall n$. If $\frac{a_{n+1}}{a_n} \to a$ & $0 \leq a < 1$ then $(a_n) \to 0$ (N.B: $a \neq 1$).*

Workbook 5

Def$^{\underline{n}}$. A real number is rational if it can be written $\frac{p}{q} s.t. p, q \in \mathbb{Z}, q \neq 0$. A real number that is not rational is irrational.

Th$^{\underline{m}}$. *$\sqrt{2}$ is irrational.*

Th$^{\underline{m}}$. *Between any two distinct real numbers there is a rational number. i.e: If $a < b \exists \frac{p}{q} \in \mathbb{Q} s.t. a < \frac{p}{q} < b$.*

Cor$^{\underline{ly}}$. *There are an infinite number of rational numbers in the interval (a, b), given $a < b$.*

Th$^{\underline{m}}$. *Between any two distinct real numbers there is an irrational number.*

Cor$^{\underline{ly}}$. *There are an infinite number of irrational numbers in the interval (a, b), given $a < b$.*

Def$^{\underline{n}}$. A- a non-empty set of real numbers, is:

- bounded above if $\exists U s.t. a \leq U \forall a \in A$ (U is an upper bound).
- bounded below if $\exists L s.t. a \geq L \forall a \in A$ (L is a lower bound).
- bounded if it is bounded above and below.

Def$^{\underline{n}}$. u is a least upper bound of A if:

- u is an upper bound of A.
- if U is any upper bound of A then $u \leq U$. ($\sup A = u$)

ℓ is a greatest lower bound of A if:

- ℓ is an lower bound of A.
- if U is any lower bound of A then $\ell \leq U$. ($\inf A = \ell$)

Completeness Axiom: Every non-empty subset of the reals that is bounded above has a least upper bound. See workbook 6 for alternative forms.

Th$^{\underline{m}}$. *Every positive real number has a unique k^{th} root.*

Workbook 6

Th$^{\underline{m}}$. *Bolzano-Weierstrass: Every bounded sequence has a convergent subsequence.*

Convergence Test: A monotonic sequence converges iff it is bounded.

Def$^{\underline{n}}$. (a_n) is Cauchy $\iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ s.t. \ \forall n, m > N, \ |a_n - a_m| < \varepsilon$.

Th$^{\underline{m}}$. *Every Cauchy sequence is convergent.*

Convergence Test: A sequence is convergent iff it has the Cauchy property.
Convergence Axiom: Every non-empty set A of the real numbers which is bounded above has a least upper bound; $\sup A$.
Equivalent Conditions:

- Every non-empty set A of the real numbers which is bounded below has a greatest lower bound; $\inf A$
- Every bounded increasing sequence is convergent.
- Every bounded decreasing sequence is convergent.
- Every bounded sequence has a convergent subsequence.
- Every infinite decimal sequence is convergent.

Def$^{\underline{n}}$. A positive real number x has a representation as an infinite decimal if there is a non-negative integer d_0 & a sequence (d_n) with $d_n \in \{0, \ldots, 9\} \ \forall n s.t.$ the sequence with n^{th} term defined by: $d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} = \sum_{j=0}^{n} d_j \cdot 10^{-j}$ converges to x. We write $x = d_0.d_1d_2d_3\ldots$

Th$^{\underline{m}}$. *Every infinite decimal $\pm d_0.d_1d_2d_3\ldots$ represents a real number.*

Th$^{\underline{m}}$. *If a positive real number has two different representations as an infinite decimal, one is finite (i.e. ends $\ldots 000 \ldots$) and the other ends with recurring nines (i.e. ends $\ldots 999 \ldots$).*

Def$^{\underline{n}}$. $\pm d_0.d_1d_2d_3\ldots$ is:

- terminating if $\exists N \in \mathbb{N} \ s.t. \ \forall n > N \ d_n = 0$.

- recurring if $\exists N, r \in \mathbb{N}$ $s.t.$ $\forall n > N$ $d_n = d_{n+r}$.

- non-recurring if it is not terminating or recurring.

Th$^{\underline{\text{m}}}$. *A number x can be represented by a terminating decimal iff $x = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ where the only prime factors of q are 2's & 5's.*

Th$^{\underline{\text{m}}}$. *Every recurring decimal represents a rational number.*

Cor$^{\underline{\text{ly}}}$. *Every recurring decimal x with repeating block length k can be written as $x = \frac{p}{q(10^k)-1}$ where the only prime factors of q are 2's & 5's.*

Th$^{\underline{\text{m}}}$. *Every rational number can be represented by a recurring or terminating decimal.*

Th$^{\underline{\text{m}}}$. *The rationals are the sent of terminating or recurring decimals. The irrationals are the set of non-recurring decimals.*

Workbook 7

Def$^{\underline{\text{n}}}$. Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ with partial sums (s_n) where $s_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i$

- $\sum_{n=1}^{\infty} a_n$ converges if (s_n) converges, if $(s_n) \to s$ then $\sum_{n=1}^{\infty} a_n = s$.
- $\sum_{n=1}^{\infty} a_n$ diverges if (s_n) does not converge.
- $\sum_{n=1}^{\infty} a_n$ diverges to $\pm\infty$ if $s_n \to \pm\infty$.

<u>Geometric Series</u>: $\sum_{n=0}^{\infty} x^n$ is convergent if $|x| < 1$ and its sum is $\frac{1}{1-x}$. It is divergent if $|x| \geq 1$.
<u>Sum rule for series</u>: If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ are convergent then $\forall c, d \in \mathbb{R}$, $\sum_{n=1}^{\infty}(ca_n + db_n)$ is convergent & $\sum_{n=1}^{\infty}(ca_n + db_n) = c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n$.
<u>Shift rule for series</u>: Let $N \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} a_{N+n}$ converges.
<u>Boundedness Condition</u>: If $a_n \geq 0$ then $\sum_{n=1}^{\infty} a_n$ converges iff $s_n = \sum_{j=1}^{n} a_j$ is bounded.
<u>Null sequence test</u>: If $(a_n) \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ diverges. (Diverges only)
<u>Comparison test</u>: If $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$ & $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges & $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Cor$^{\underline{\text{ly}}}$. *If $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$ & $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.*

Def$^{\underline{\text{n}}}$. $e := \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Th$^{\underline{\text{m}}}$. $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$

Th$^{\underline{\text{m}}}$. *e is irrational.*

Workbook 8

<u>Ratio test</u>: If $a_n > 0$ $\forall n \in \mathbb{N}$ & $\frac{a_{n+1}}{a_n} \to \ell$ then $\sum a_n$ converges if $0 \leq \ell < 1$ & diverges if $\ell > 1$.
<u>Integral test</u>: If $f(x) > 0$ & decreasing for $x \geq 1$ then:

- $\sum_{n=1}^{\infty} f(n)$ converges if the sequence $(\int_1^n f(x).dx)$ is bounded.
- $\sum_{n=1}^{\infty} f(n)$ diverges if the sequence $(\int_1^n f(x).dx)$ is unbounded.

Cor$^{\underline{\text{ly}}}$. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges for $p > 1$ & diverges for $0 < p \leq 1$.*

Workbook 9

<u>Alternating series test</u>: If (a_n) is decreasing & null then $\sum(-1)^{n+1}a_n$ is convergent.

Def$^{\underline{\text{n}}}$. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

Th$^{\underline{\text{m}}}$. *Every absolutely convergent series is convergent. (N.B: not vice versa)*

<u>Ratio test</u>: If $a_n \neq 0$ $\forall n \in \mathbb{N}$ & $\left|\frac{a_{n+1}}{a_n}\right| \to \ell$ then $\sum a_n$ converges absolutely (hence converges) if $0 \leq \ell < 1$ & diverges if $\ell > 1$.
<u>Ratio test varient</u>: If $a_n \neq 0$ $\forall n \in \mathbb{N}$ & $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ then $\sum a_n$ diverges.

Workbook 10

Def$^{\underline{n}}$. (b_n) is a rearrangement of (a_n) if ther exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ (i.e: a permutation on $\mathbb{N}$) s.t. $b_n = a_{\sigma(n)}$ $\forall n$.

Lemma. *is $\sum a_n$ is a convergent series of non-negative terms & (b_n) is a rearrangement of (a_n) then $\sum b_n$ converges & $\sum b_n = \sum a_n$.*

Th$^{\underline{m}}$. *If $\sum a_n$ is an absolutely convergent series & (b_n) is a rearrangement of (a_n) then $\sum b_n$ then $\sum b_n$ is convergent & $\sum b_n = \sum a_n$.*

Def$^{\underline{n}}$. $\sum a_n$ is conditionally convergent if $\sum a_n$ is convergent but $\sum |a_n|$ is not.

Lemma. *If a series is conditionally convergent the series of its positive terms diverges to infinity & the series of its negative terms diverges to minus infinity.*

Th$^{\underline{m}}$. *Rieman's Rearrangement: If $\sum a_n$ is conditionally convergent then $\forall \ell \in \mathbb{R}$ $\exists (b_n)$ a rearrangement of (a_n) s.t. $\sum b_n = \ell$.*